MATHEMATICAL GAMES

Mathematical strategies for two-person contests

by Martin Gardner

The word "games" in the title of this department has a broad meaning synonymous with "recreations," but from time to time the monthly topic has been games in the narrower sense of actual contests between players. This month we consider a variety of two-person games, some old and some new, for which mathematical strategies are known. First, here is a trio of simple games that are related to each other in an amusing and surprising way.

1. Nine playing cards, with values from ace to nine, are face up on the table. Players take turns picking a card. The first to obtain three cards that add to 15 is the winner.

2. On the road map at the bottom of the opposite page players take turns eliminating one of the nine numbered highways. This is done by coloring the complete length of the road, even though it may go through one or two towns (*circles*). Pencils of two different colors are used to distinguish the moves of the two players. The first to color three highways that enter the same town is the winner. (The Dutch psychologist John A. Michon, who invented this game, calls it "Jam" because those are his initials and because the object of the game is to jam crossings by blocking highways.)

3. Each of the following words is printed on a card: HOT, HEAR, TIED, FORM, WASP, BRIM, TANK, SHIP, WOES. The nine cards are placed face up on the table. Players take turns removing a card. The first to hold three cards that bear the same letter is the winner. (The Canadian mathematician Leo Moser, who devised this game, calls it "Hot.")

For each game the question is: If both players make their best moves, is the game a win for the first player, a win for the second player or a draw? Perhaps the reader has already experienced what the Gestalt psychologists call "closure" and recognized that all three games are isomorphic with ticktacktoe!

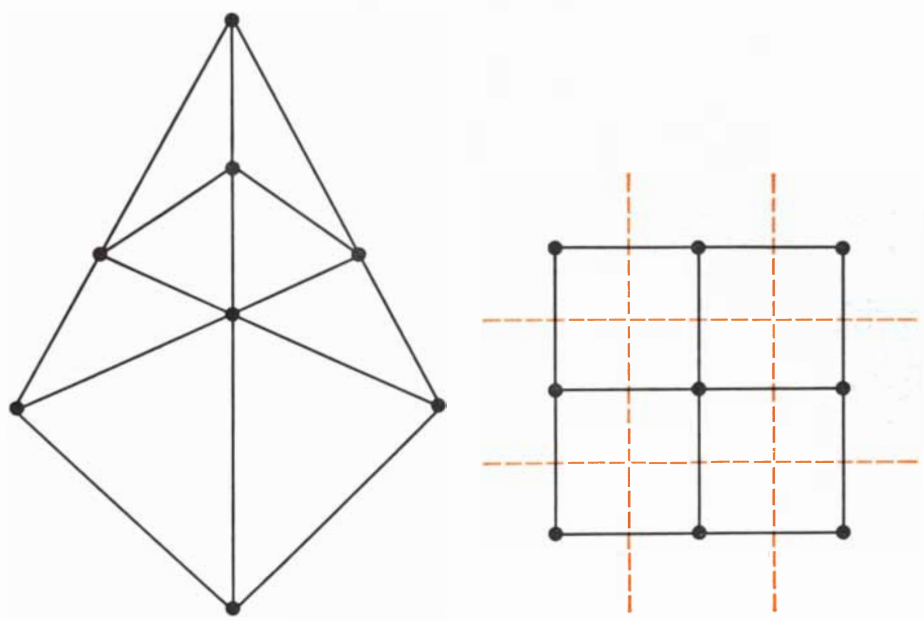
It is easy to see that this is the case. For the first game we make a list of all the triplets of distinct digits from 1 to 9 that have a sum of 15. There are exactly nine such triplets. They can be interlocked on a ticktacktoe board as shown at the top of the opposite page to form the familiar order-3 magic square on which every row, column and main diagonal is one of the triplets. Each numbered card drawn by a player corresponds to a ticktacktoe play on the cell of the magic square that bears that digit. Each set of triplets that wins in the card game corresponds to a winning ticktacktoe row on the magic square. Anyone who can play a perfect game of ticktacktoe and who also memorizes the magic square can immediately play a perfect game in this card version.

The map at the bottom of the opposite page is topologically equivalent to the symmetrical graph shown at the left in the illustration on this page. This is in turn the "dual" of the graph obtained by connecting the centers of the nine cells of a ticktacktoe board as shown at the right in the illustration. Each numbered cell of the magic square corresponds to a numbered highway on the map and each town on the map corresponds to a row, column or main diagonal on the magic square. As before, there is an equivalence relation between plays on the map and plays in ticktacktoe.

The isomorphism of Moser's word game and ticktacktoe becomes obvious when the nine words are written inside the cells of a ticktacktoe matrix as shown at the top of page 118. Each set of three-in-a-row words has a common letter, and there are no such sets other than the nine displayed in this way. Again, memorizing the square of words instantly enables a perfect-game ticktacktoe player to play a perfect game of Hot. Since ticktacktoe played rationally is always a draw, the same is true of the three equivalent games, although the first player naturally has a strong advantage over a second player who is not aware that he is playing disguised ticktacktoe or who may not play a perfect game of ticktacktoe.

One who grasps the essential identity of the three games will have obtained a valuable insight; mathematics abounds with "games" that seem to have little in common and yet are merely two different sets of symbols and rules for playing the *same* game. Geometry and algebra, for example, are two ways of playing exactly the same game, as Descartes's great discovery of analytic geometry shows.

There are many games of the "take away" type in which players alternately



Graph of Jam map (left) *and its ticktacktoe "dual"* (right) take away an element or subset from a set, the winner being the person who acquires the last element. The best-known game of this kind is nim, played with a set of counters arranged in an arbitrary number of rows, with an arbitrary number of counters in each row. On his turn a player may take as many counters as he wishes, provided that they all come from the same row. The person who takes the last counter wins. A perfect strategy is easily formulated in the binary system, as explained in *The Scientific American Book of Mathematical Puzzles & Diversions.*

A starting pattern for nim, as it was played throughout the 1962 French movie *Last Year at Marienbad,* is shown at the bottom of the next page. Sixteen cards are arranged in four rows of one, three, five and seven cards. (The triangular pattern symbolizes the triangular love game played in the picture.) To determine whether the first or the second player can win we write the numbers of cards in each row in the binary system, then add the columns:

1	1
3	11
5	101
7	111
	224

If the sum of every column is an even number (or zero if the addition is made modulo 2), as in this case, the pattern is called "safe." This means that the first player is certain to lose against an expert, for regardless of how he plays he will leave an "unsafe" pattern (one with at least one column that has an odd sum), and the second player can convert this to another safe position on his next move. By always playing to leave a safe pattern he is sure to get the last counter.

Michel Hénon, a mathematician at the Centre National de la Recherche Scientifique in Paris, recently thought of a delightful nim variant, played with scissors and pieces of string. It is best approached, however, by first explaining an older variant of nim called kayles, to which the string game is closely linked.

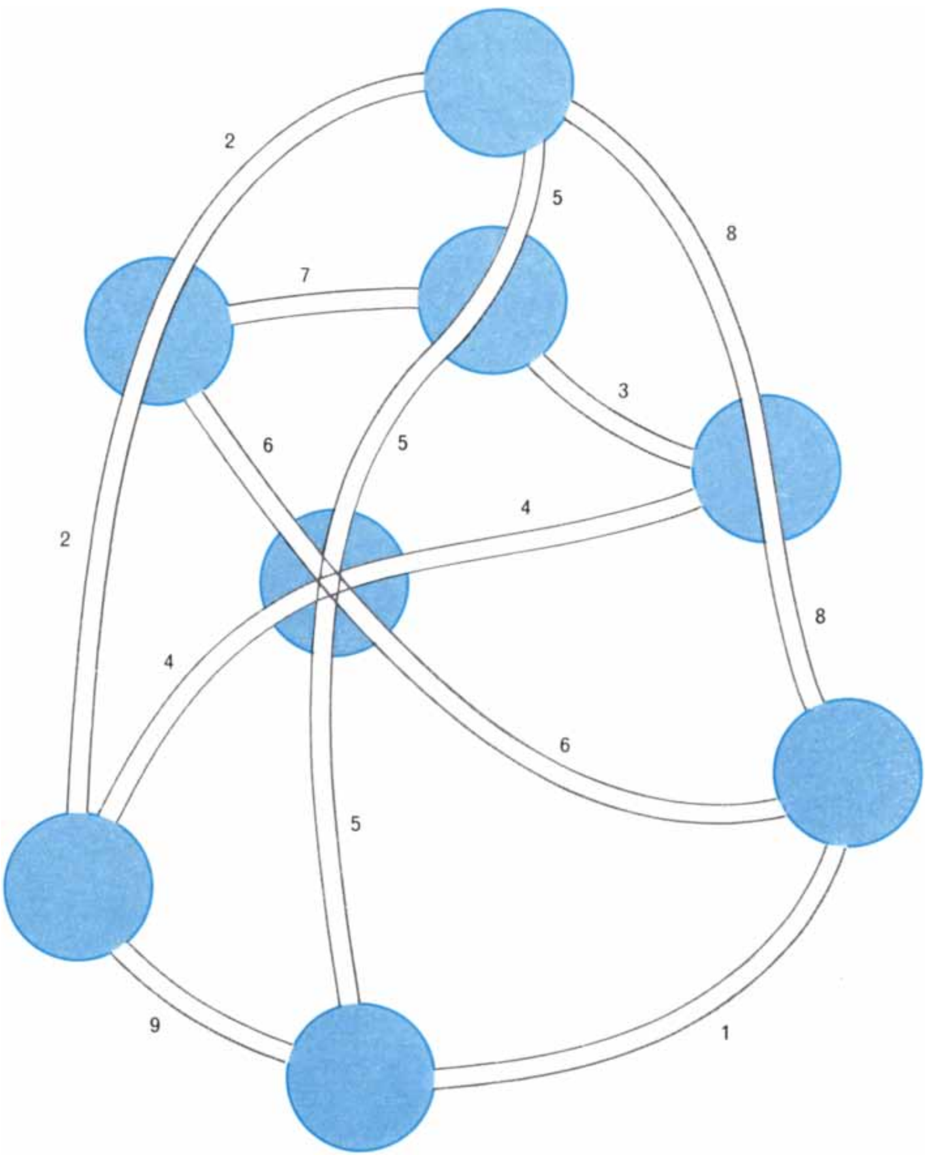
Kayles was invented by the English puzzle expert Henry Ernest Dudeney, who introduced it in Problem 73 of his first book, *The Canterbury Puzzles* (1907). It is now called kayles because Dudeney presented it as a problem that might have arisen in playing a popular 14th-century game of that name in which a ball was rolled at wooden pins standing side by side. The ball's size was such that it could knock over either a single pin or two touching pins. Mathematical kayles is best played on the table with coins, cards or other objects simply by arranging them in an arbitrary number of rows, exactly as in nim, with an arbitrary number of objects in each row. Now, however, we must think of each row as a linked chain. One may remove one link or two adjacent links. If the object or pair of objects is taken from inside a chain, it breaks the chain into two separate chains. For instance, if the first player takes the center card from the bottom row of the Marienbad pattern, it breaks the seven cards into two separate chains of three links each. In this way the number of chains is likely to increase as the game continues.

Kayles also lends itself to binary analysis, but not as directly as nim. For every chain we associate a binary number,

2	9	4
7	5	3
6	1	8

Ticktacktoe version of card game

but that number (except for the three smallest cases) is not the same as the decimal number of the cards in the chain. The chart shown at the bottom of page 119, supplied by Hénon, gives the required binary number, here called the *k* number, for integers 1 through 70. After 70 a curious periodicity of 12 num-



Map for the game of Jam

HOT	FORM	WOES
TANK	HEAR	WASP
TIED	BRIM	SHIP

Key to the game of Hot

bers sets in. If the number is above 70, divide it by 12, note the remainder, then use the chart at the bottom right of the illustration. To decide if a kayles pattern is safe or unsafe, one uses k numbers like the nim binary numbers.

Consider the Marienbad starting position, which is safe in nim and therefore a win for the second player. Is it also safe in kayles? Using k numbers we find:

1	1
3	11
5	100
7	10
	122

The sums are not all even, so the position is unsafe in kayles. Only one move by the first player will create a safe pattern, thereby ensuring a win. Can the reader discover it?

The derivation of the k numbers is too complicated to explain here. The interested reader will find it detailed by R. K. Guy and C. A. B. Smith in *Proceedings of the Cambridge Philosophical Society,* Vol. 52, 1956, pages 516–526, and by Thomas H. O'Beirne in *Puzzles and Paradoxes,* Oxford University Press, 1965, pages 165–167. Note that no k number has more than four digits. As a result there are 16 different four-term combinations of odd and even that can occur as column sums, only one of which is even-even-even-even. As Hénon points out, this enables us to conclude, with a high degree of accuracy, that if a kayles starting position is chosen at random from all possible patterns, the probability is extremely close to 1/16 that it will be safe.

There are helpful rules that a kayles player can follow without having to analyze each pattern. Two equal chains are safe because whatever your opponent does to one you can do exactly the same to the other. For example, if the two chains are 5 and 5 and he takes the second card in one, you take the second in the other. This leaves chains of 1, 1, 3, 3. If he takes two cards from a 3-chain, you take two from its twin. If he takes a 1-card chain, you take the other. It follows that if the starting position is one single chain, the first player has an easy win. If the chain has one or two cards, he takes them. If it has more than two cards, he takes one or two from the center to leave two equal chains and then continues as explained. If a pattern has an even number of equal pairs of chains, the position is clearly safe, since whatever the first player does to one chain the second player does to that chain's twin.

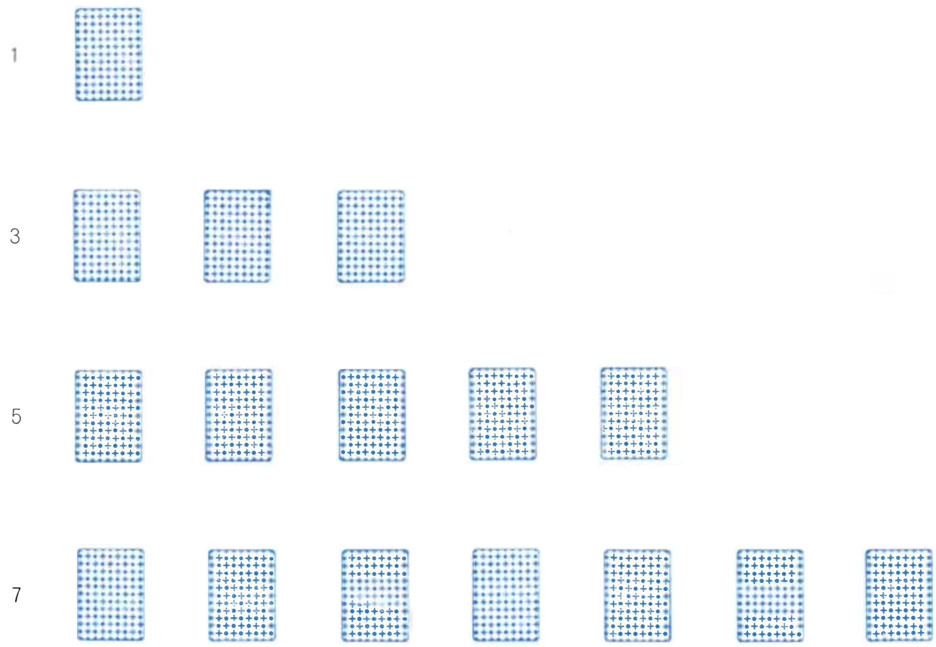
It is also good to remember the following safe patterns for two or three chains with no more than nine cards in each. The safe doublets (aside from two equal chains, which are always safe) are 1–4, 1–8, 2–7, 3–6, 4–8 and 5–9. Safe triplets can be calculated in the head by memorizing the following three groups: 1, 4, 8; 2, 7 and 3, 6. Any triplet made up of one digit from each group is safe.

Let us turn now to Hénon's two-person string game. We are given an arbitrary number of pieces of string of arbitrary lengths. Players take turns cutting a one-inch segment from any piece of string. The segment can be cut from the end or it can be cut, with two snips, from the interior. In the second case it will leave two pieces of string where there was one before. A one-inch piece can, of course, be taken without any snipping. The person who gets the last one-inch piece wins.

String lengths need not be rational. In the top illustration on the opposite page a game begins with four strings of lengths 1, pi, the square root of 30 and the square root of 50. Who has the win if both sides play rationally? This seems at first an enormously difficult question, but with the proper insight it is absurdly easy. To work on the problem rule four straight lines of about the required lengths. As each one-inch segment is erased the remaining lines are labeled with correct lengths.

The game can also be played with closed loops of string. Suppose it begins with seven such loops, each with a length greater than two inches. Without knowing any of the actual lengths, which player has the win? Approached in the right way, this is even easier to answer than the previous question.

Our final game, called the "game of the hamstrung squad car," is taken from Rufus Isaacs' book *Differential Games* (Wiley, 1965). Devotees of recreational mathematics may recall that Isaacs provided the excellent illustrations for the late James R. Newman's popular *Mathematics and the Imagination,* but among mathematicians Isaacs is best known as an operations-research expert. He is now at the Center for Naval Analyses in Washington. Last year his book shared the Lanchester Prize, given annually by the Operations Research Society of America for outstanding contributions in the field. The book is filled with original methods of solving difficult conflict games of the kind often encountered in military situations, particularly games that have to do with pursuit and capture. Some of these games are discussed in simpler, discrete ver-



Marienbad starting pattern for nim or kayles sions that have great recreational interest.

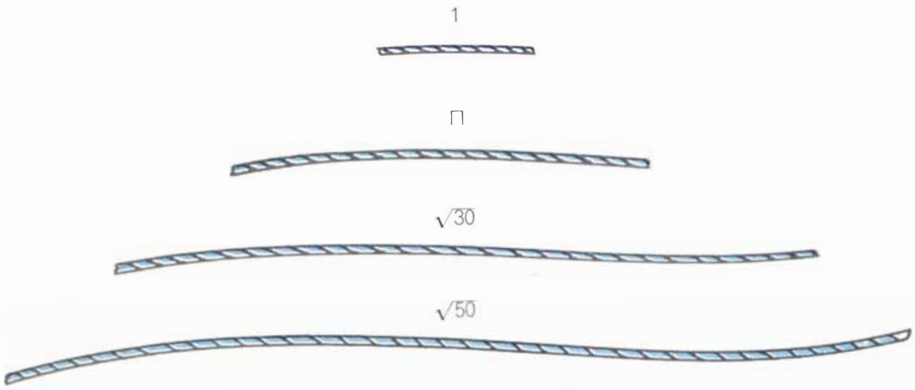
One of the book's key games, completely solved by Isaacs, is what he calls the "homicidal-chauffeur game." Imagine a homicidal chauffeur at the wheel of a car he is driving on an infinite plane. He moves at a fixed speed. He can shift the position of his steering wheel instantaneously, but the degree to which he can turn the front wheels is limited. Also on the infinite plane is a lone pedestrian. He can move in any direction at any instant. His speed too is constant, but less than the car's speed. Under what conditions can the car (assumed to be a positive area surrounding the driver) always catch (touch) the pedestrian? Under what conditions can the pedestrian escape permanently? How can the pursuer minimize the time it takes to run down his quarry when this is possible?

Fortunately we shall not be concerned with these difficult questions but with the simpler and somewhat similar game Isaacs calls the "hamstrung squad car." Imagine a city of infinite extent, with streets that form a regular square lattice. A squad car is at one intersection. At another is a carful of criminals. The squad car moves twice as fast as the criminals' car but is hamstrung by having to observe municipal traffic rules that prohibit left turns and U-turns, so that it can only go straight ahead or turn right at each intersection; the criminals' car does not observe these restrictions, so that at each intersection it can move in any of the four directions.

For the quantized game, intersections are replaced by the squares of an infinite checkerboard. The squad car is a counter with a vector arrow painted on it to indicate the direction in which it is moving. The criminals' car is an unmarked counter. Players take turns, the squad car making the first move. All moves are like rook moves in chess: up, down, left or right but never diagonally. The criminals move one square at a time. The squad car moves two squares, always in a straight line, either in the direction it has been traveling or after making a right turn. (It cannot go one square, turn right and then go another.) It "captures" the criminals if it lands on the square occupied by them or on a square that is adjacent orthogonally or diagonally.

These rules are illustrated on the next page. The squad car can move to squares A or B on its first move. From A it can then move to C or D; from B, to E or F. After each move it should be turned (if necessary) so that its arrow shows the direction in which it was last moving. The criminals can move to squares W, X, Y or Z. If the squad car were on F and the criminals were on the same square or on any of the eight shaded squares surrounding it, they would be considered caught.

From what starting positions can the criminals be captured? Isaacs shows that surrounding the squad car's initial square there is an asymmetrical, compact area of exactly 69 squares each



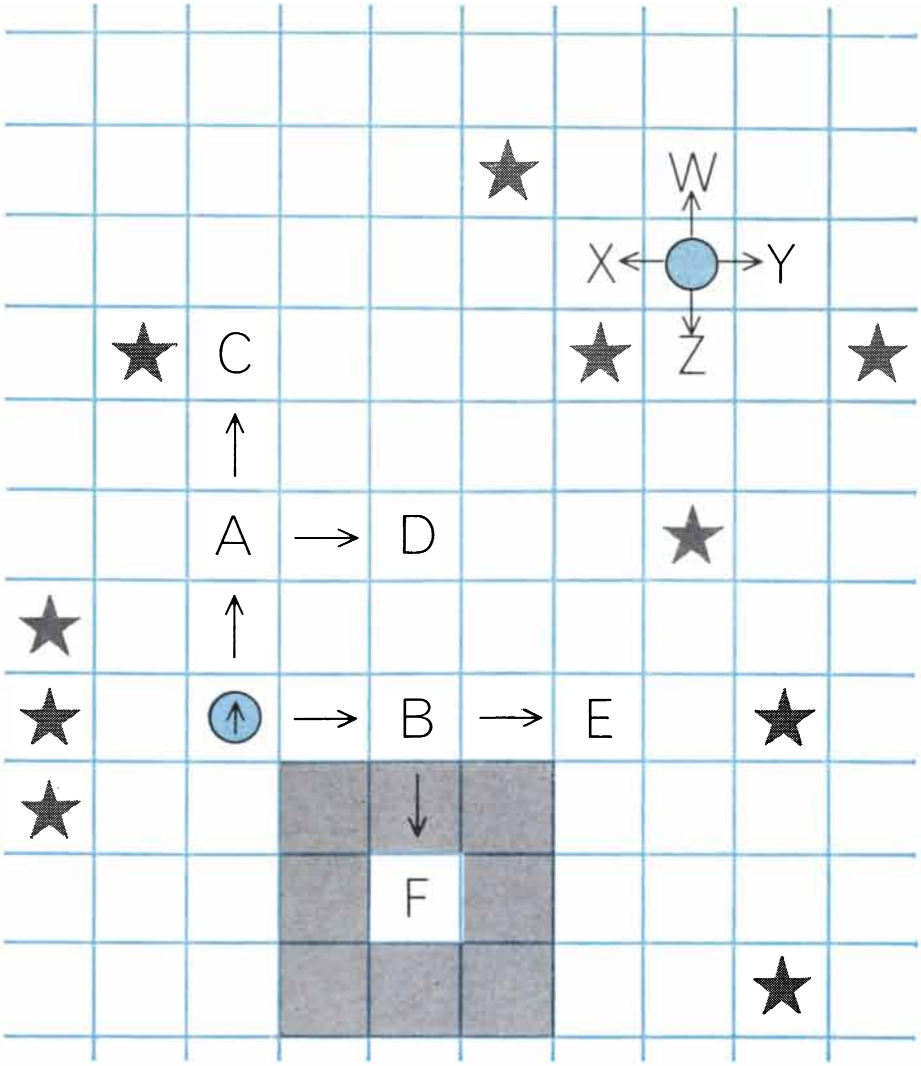
Hénon's string game

NUMBER IN ROW	K NUMBER	NUMBER IN ROW	K NUMBER	NUMBER IN ROW	K NUMBER
1	1	31	10	61	1
2	10	32	1	62	10
3	11	33	1000	63	1000
4	1	34	110	64	1
5	100	35	111	65	100
6	11	36	100	66	111
7	10	37	1	67	10
8	1	38	10	68	1
9	100	39	111	69	1000
10	10	40	1	70	110
11	110	41	100		
12	100	42	111		
13	1	43	10		
14	10	44	1		
15	111	45	1000		
16	1	46	10		
17	100	47	111		
18	11	48	100		
19	10	49	1		
20	1	50	10		
21	100	51	1000		
22	110	52	1		
23	111	53	100		
24	100	54	111		
25	1	55	10		
26	10	56	1		
27	1000	57	100		
28	101	58	10		
29	100	59	111		
30	111	60	100		

NUMBERS OVER 70

REMAINDER	K NUMBER
0	100
1	1
2	10
3	1000
4	1
5	100
6	111
7	10
8	1
9	1000
10	10
11	111

Binary k numbers for playing kayles



Isaacs' "hamstrung squad car" game

of which is a fatal starting spot for the criminals. If they start on any square outside this area they can (assuming an unbounded board) always escape permanently.

The reader is urged to draw a large checkerboard of, say, 50 squares by 50 (or find a room with a suitable floor pattern), choose a starting position near the center for the squad car and see if he can identify the 69 fatal squares. Until the game is fully analyzed it can provide much amusement. The player moving the criminals' car can choose his starting spot and then see if he can win by reaching the border before he is caught. Playing the game long enough will eventually outline the fatal area, but there is a simpler way by which it can be delineated quickly and each of its squares given a number indicating the number of squad car moves required for the capture if both sides play rationally.

For readers not inclined to make a complete analysis here is a simpler problem. Assume that the squad car starts from the position shown in the illustration. The criminals may start from any of the 10 starred squares. From all but one of these starred positions they can escape permanently. Which is the fatal starred square, and in how many moves will the criminals be caught if they start from that square and both sides make their best moves? These and the other questions will be answered in the next issue.

The acrostic sonnet given last month is by Edgar Allan Poe. He called it "An Enigma." The lady's name, Sarah Anna Lewis, is read by taking the first letter of the first line, the second letter of the second line and so on through the 14 lines. The poem is not so well known as Poe's other acrostic, "A Valentine," in which another lady's name is similarly concealed. In J. A. Lindon's poem the first line is repeated not only by the first words of the eight lines but also by the *n*th words of the *n*th lines.