MATHEMATICAL GAMES

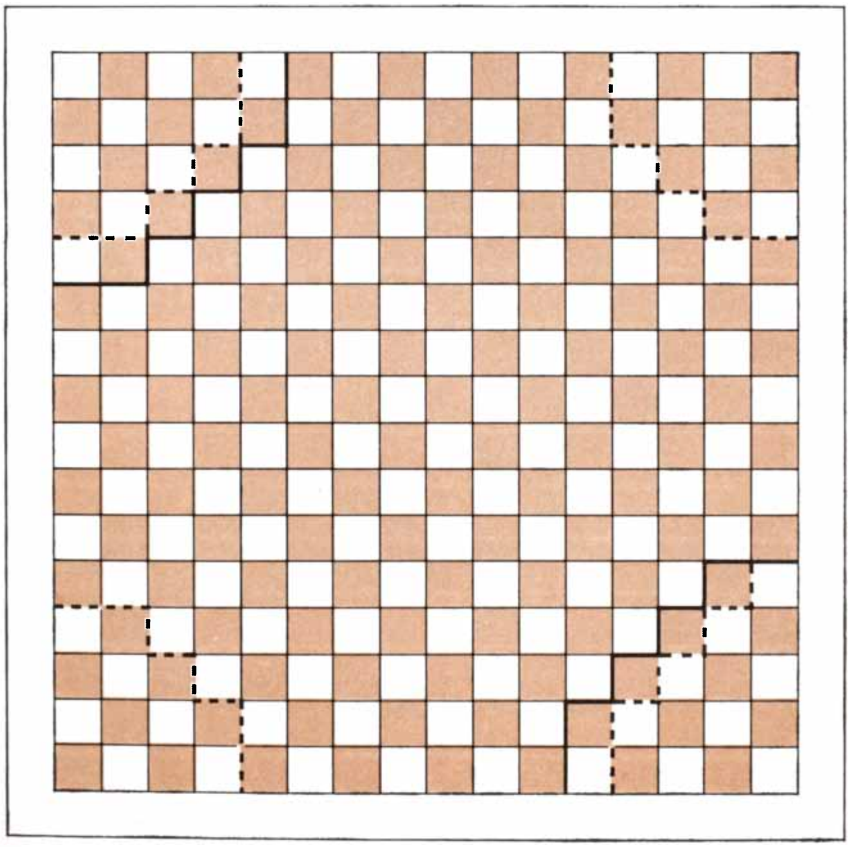
New puzzles from the game of Halma, the noble ancestor of Chinese checkers

by Martin Gardner

Two new families of puzzles based on a long-neglected counter-moving game have recently come to light. Each family of puzzles offers a series of unsolved problems and the opportunity to devise ingenious proofs that some solutions are impossible. The puzzles stem from *Dialogue on Puzzles*, a splendid collection of unusual problems by Kobon Fujimura and Michio Matsuda published this year in Japan. (Unfortunately the book is not available in English.) Fujimura has translated the puzzle books of Sam Loyd and Henry Ernest Dudeney into Japanese and is the author of several delightful books that contain his own original puzzles. The two new counter-moving puzzles are derived from one problem created by Matsuda.

Matsuda's problem exploits the simple rules of a popular late-19th-century British proprietary game called Halma, after the Greek word for leap. The game was invented in 1883 by George Howard Monks, a 30-year-old Harvard Medical School graduate who was then pursuing advanced studies in London. He later became a prominent Boston surgeon. Halma is still played in Britain but, although it was issued here in 1938 by Parker Brothers, it has never caught on in this country.

The traditional Halma board has 16 cells on a side [*see illustration below*].



The Halma board

If two players are competing, each begins by placing his 19 counters in a section called a "yard." There are two yards, one at the top left corner of the board and the other at the bottom right corner. The counters are identical except that the two sets are of contrasting colors. The goal is to occupy the opposing player's yard, and the first player to move all his counters into the opposite yard is the winner. Two kinds of move are allowed: (1) A "step." This is a move, like the move of a chess king, to any one of the eight adjoining cells. (2) A "hop." This is a leap over another counter, as in checkers, except that the leap may be made in any direction, either orthogonal or diagonal. The jumped piece is not removed.

A connected chain of hops counts as a single move. It is not compulsory to make a hop. A player may continue a chain of jumps as long as possible or stop wherever he pleases. The color of a jumped piece does not matter; a chain of jumps may be a mixture of friendly and enemy counters. Steps and hops may not, however, be combined in the same move. Players alternate turns, moving one counter at a time.

Halma can also be played by four people, with each player having 13 counters. The yards are at the four corners of the board behind the boundaries indicated by the broken lines in the illustration. The four-player game can be each man for himself, with each seeking to reach the diagonally opposite yard, or pairs of opposite (or adjacent) players can be partners who help each other, and the first pair to yard all 26 of their counters is the winner. Halma strategy is so complex, however, that the game is best when only two people play.

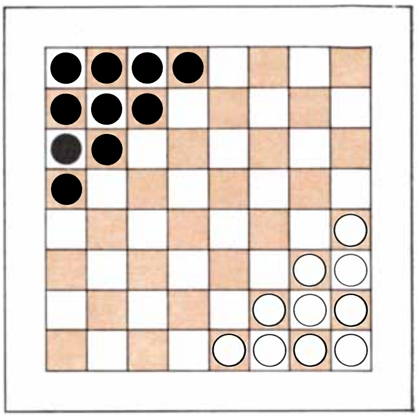
Of many later games based on Halma the two most popular in the U.S. have been Camelot and Chinese checkers, both of which appeared on the market in the 1930's. Camelot was a revival (with minor changes) by Parker Brothers of a late-19th-century Parker game called Chivalry. Chinese checkers, which has no connection whatever with China, is played on a hexagonal-cell board that is usually shaped like a six-pointed star. The hexagonal tessellation allows steps and hops in only six directions. A French version of Halma, known as Grasshopper, can be played on a standard checkerboard [*see top illustration on opposite page*]. It is an excellent game.

To prevent a stubborn player in games of the Halma type from forcing a draw by keeping a man permanently in his own yard it is wise to add extra rules.

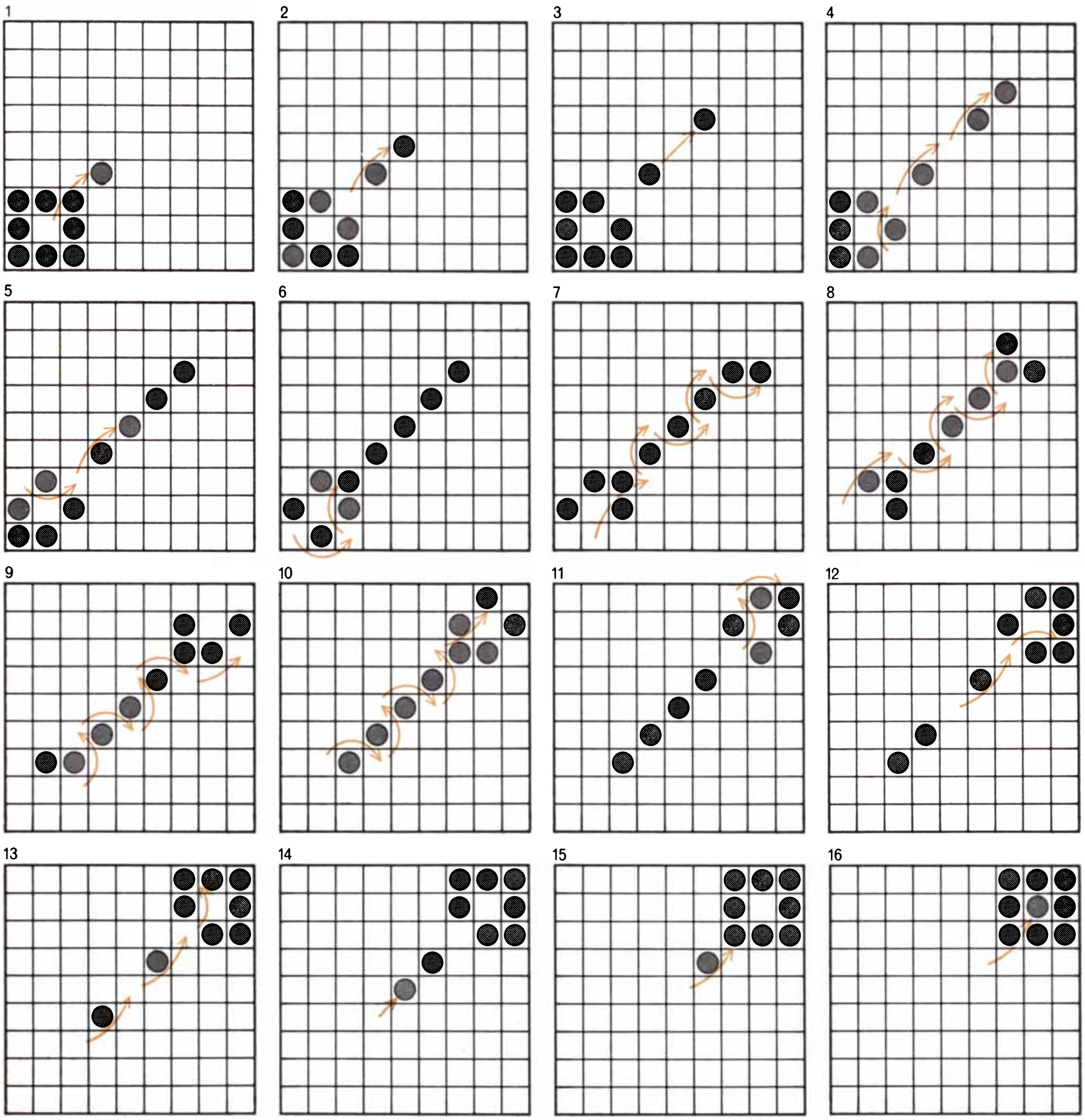
Sidney Sackson, the New York City game inventor and game collector, suggests the following. If a counter can leave its own yard by jumping an enemy counter, or by a chain of jumps that starts with a leap over an enemy counter, it must do so, although once out of the yard it may stop jumping at any desired spot. After a counter has left its yard it may not rest in the yard again, although it may hop across it.

The Halma problem devised by Matsuda for the Japanese chessboard, which has nine cells on a side, begins with nine counters in a square array at the board's lower left corner. How few moves of the Halma type, Matsuda asked himself, are needed to transfer the nine counters to the same formation at the upper right corner? He found a solution in 17 moves, but this was reduced to 16 moves [*see illustration below*] by H. Ajisawa and T. Maruyama. The 16-move solution is believed to be the minimum one.

When I saw this elegant solution, I at once began tackling the same problem on the Western chessboard with eight cells on a side and on smaller square boards with seven and six cells on a side. Using the technique of first establishing a diagonal ladder—a basic strategy, by the way, of all games of the



Grasshopper



Solution to Matsuda's problem on the Japanese chessboard

Halma type—the best I could achieve was 15 for the chessboard, 13 for the order-7 and 12 for the order-6. I have been unable to prove that any of these are minimum solutions. It is not hard to show that at least 12 moves are necessary for the order-8 square, 10 for the order-7 and 11 for the order-6.

Next I experimented with a similar transfer of the nine counters, on the same three boards, to the lower right corner instead of the corner diagonally opposite. The order-6 board has many solutions in nine moves, one of which is shown in the illustration below. Nine is obviously minimal because each counter must move once. (It is necessary that at least one counter hop to and from the fourth row on its way to the other yard, consequently nine-move solutions cannot be achieved on a three-by-six board.) On the order-7 board 10 moves will do it. This too is readily seen to be minimal since the first piece to move must move at least once again to reach the adjacent yard.

Thirteen moves will solve the problem on the order-8 board. That 12 are necessary is evident from a simple parity check. The six counters in column 1 and column 3 can hop only to column 7, therefore three of the six must each make at least one step move. I tried vainly for weeks to find a 12-move solution until Donald E. Knuth, a mathematician at Stanford University, came to my rescue by devising a proof of impossibility in 12 moves. It is too involved to give here, but it is based on the necessity for one of the original four corner counters to step to a different color, the fact that the reverse of a solution is another solution and other considerations. Readers may enjoy searching for minimum solutions to the six transfer problems. None will be given next month. But if someone succeeds in lowering the smallest number of moves known for one of three diagonal-transfer problems or manages to prove that the numbers given are indeed minimal, I shall report the results in a later column. (If there are too many letters, I shall probably be unable to reply except by summarizing them in this department.)

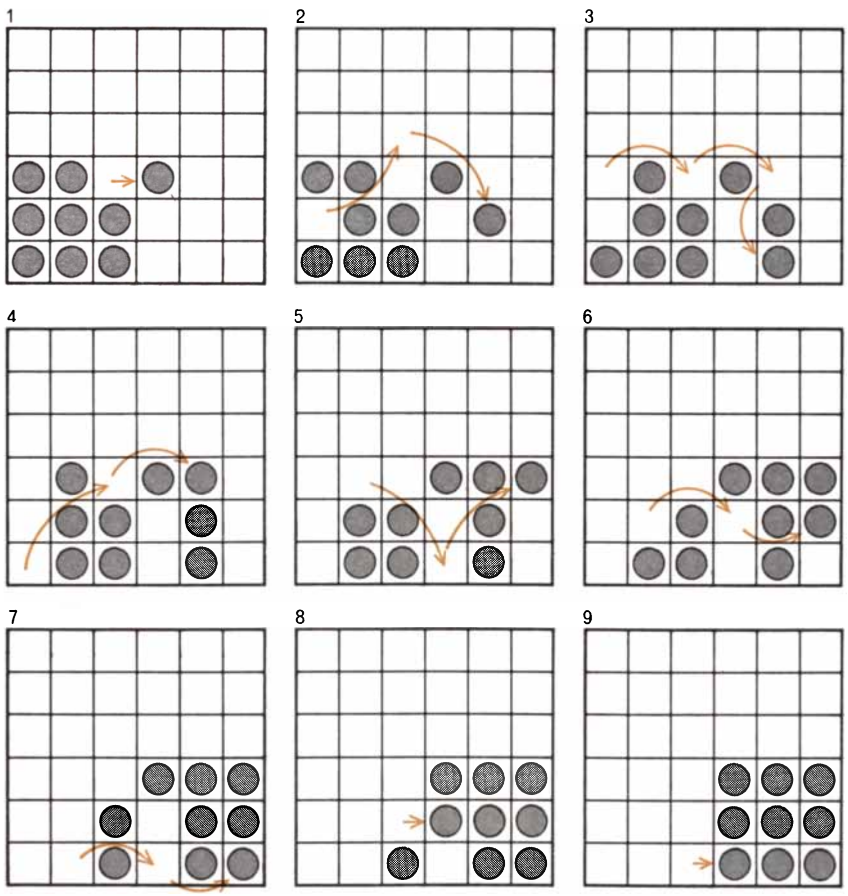
The second family of puzzles suggested by Matsuda's problem is devised by removing every jumped counter from the board. The goal is to remove all counters but one, the last counter reaching a specified cell, and do it in a minimum number of Halma moves. Such problems are similar to those of the classic peg-solitaire game discussed in an earlier column (reprinted in my book *Unexpected Hanging and Other Mathematical Diversions*) except that the greater freedom of movement allows for much shorter solutions, and proofs of minimum solutions are usually much more difficult.

Consider, for example, the puzzle on a five-square board that was first issued in 1908 by Sam Loyd [*see "a" in illustration on opposite page*]. He labeled each counter with the name of a hopeful in that year's presidential election. The idea was to eliminate eight men, leaving one's favorite on the center cell. Loyd allowed Halma moves but did not count a chain of jumps as being one move. Eight jumps are clearly minimal and there are many such solutions for each counter. Henry Ernest Dudeney, in his *Amusements in Mathematics* (Problem 229), improved the puzzle by disallowing step moves, counting jump chains as single moves and allowing any counter to end at the center. He gave a four-move solution that is surely minimal, although I know of no proof. Counter 5 jumps 8, 9, 3, 1; counter 7 jumps 4; 6 jumps 2 and 7; then 5 returns to its original cell by leaping 6.

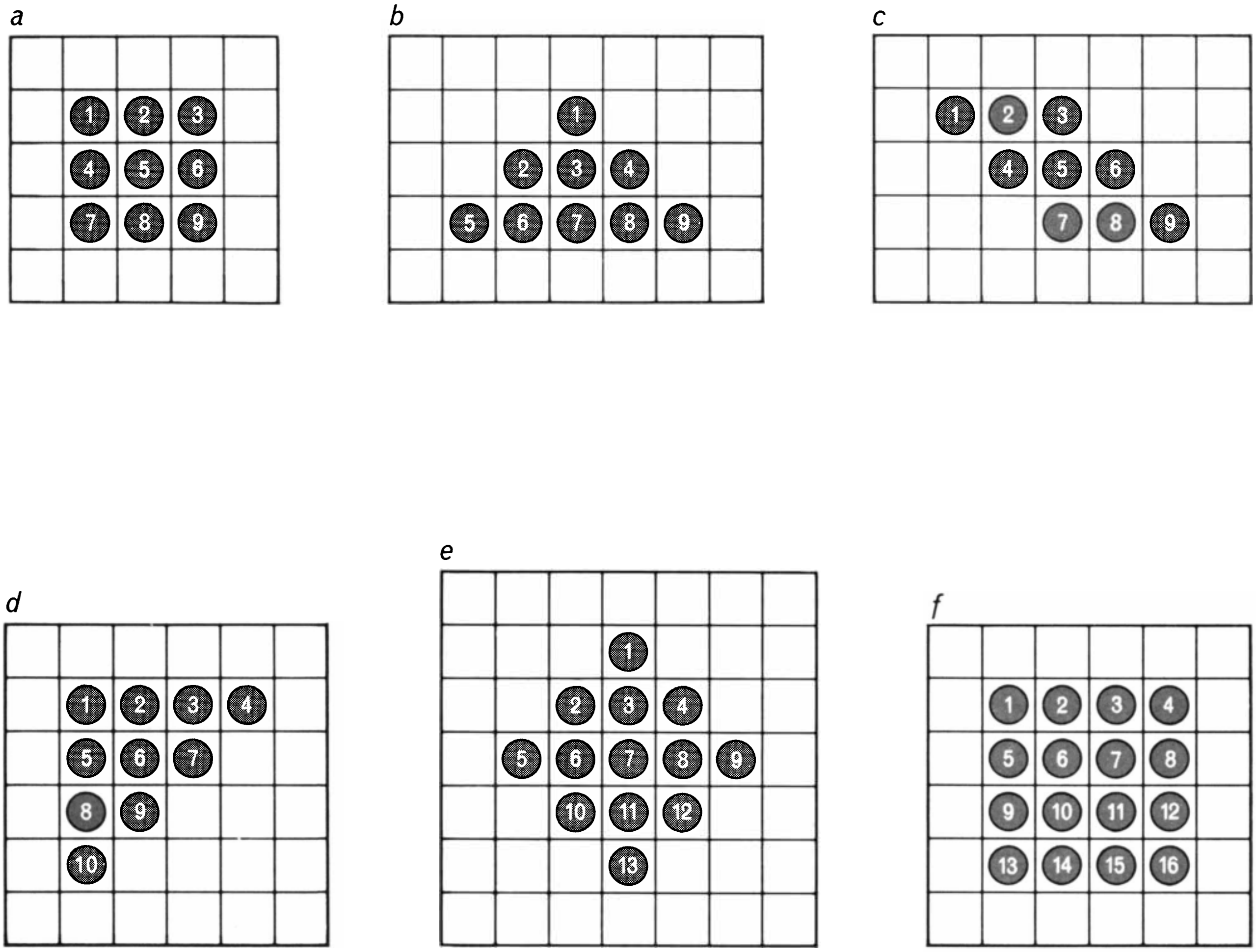
Let us combine the rules of the two rival puzzlists by allowing both steps and hops, as in Halma, and counting a chain of hops as one move. Each hopped counter is of course removed. Can the reader find one of the many three-move solutions that leave the last counter on the center cell? The solution to be given next month is an elegant one that begins with two step moves and ends with an eight-jump chain.

Similar problems are found in *b, c, d, e* and *f*. All will be answered next month: *b* is to be solved in three Halma moves, the surviving counter on the center cell; *c*, in three moves, last counter on the cell occupied by counter 1; *d*, in a minimum number of moves, last counter on the triangle's center cell, and *e* in three moves, surviving counter at the center. Problem *f*, the most difficult of the lot, calls for three moves that end with the lone counter on one of the four center cells.

I shall eventually report on any better solutions readers may find for these six problems, or on any new Halma-type problems of special interest. The field is so unexplored that it is a challenge to



Orthogonal transfer on an order-6 board



Six Halma solitaire problems

devise and solve new puzzles and then see if one can prove by simple arguments that the solution actually is minimal. I have not the slightest notion, for example, how few moves are required on an order-7 board with 25 counters in a square array in the center to leave the last counter on the center cell. I have avoided trying this problem for fear of accomplishing no other work for the next month or so.

The answers to last month's problems are as follows:

1. The simplest nonconvex polyhedron with unit-square faces is the 30-face solid formed by attaching a unit cube to each face of a unit cube. Jean J. Pedersen (whose methods of braiding solids with straight strips were last month's topic) found a way to braid this solid with three strips, each crossing once diagonally over every face of the solid.

2. A regular tetrahedron can be colored with four colors only in two ways, each a mirror reflection of the other. The simple formula that applies to all five Platonic solids is to divide the factorial of the number of faces by twice the number of edges. For example, the cube can be colored with six colors in $6!/24 = 30$ ways, the octahedron with eight colors in $8!/24 = 168$ ways, and so on.

3. A cube can be colored with three colors, each color going on two faces, in six ways: one with all pairs of opposite faces alike, two ways that are mirror images with all like colors on adjacent pairs of faces, and three ways with just one pair of opposite faces alike. Only the first three ways can be plaited with three five-square straight strips in the manner explained last month.

I was mistaken in the August issue when I reported that John L. Selfridge had a proof of his assertion about the "fork" rule in go-moku. The assertion, he has since informed me, remains a conjecture. He reports, however, that a solution *has* been found for his "four-by-infinity" ticktacktoe. This is played on a strip that is four cells high and infinitely wide, the winner being the first to get four of his marks in an orthogonal or diagonal row. Carlyle Lustenberger, in his master's thesis in computer science at Pennsylvania State University, developed a computer program with a winning strategy for the first player on a four-by-30 board. The actual lower bound for the width is a few cells shorter, but I have not yet obtained the details.

The three-by-infinity board is a trivial win for the first player on his third move; indeed, the same win can be achieved if only one cell is added to the side or corner cell of the traditional order-3 ticktacktoe field. The five-by-infinity board remains unsolved. If a win for the first player could be found on this board, it would, of course, solve the go-moku game when it is played on an arbitrarily large square, with no restrictive rules.

Next month I shall discuss reader corrections and comments on the quickie problems of July, at least one of which (the Pentagon problem) was incorrectly answered.