MATHEMATICAL GAMES

Up-and-down elevator games and Piet Hein's mechanical puzzles

by Martin Gardner

Elevators, unlike cars, trains, planes, ships and other common modes of transportation, have been unduly neglected by recreational mathematicians. This month we undertake to rectify the situation by considering four unusual elevator problems. The first three were provided by Donald E. Knuth, a computer scientist at Stanford University and author of a classic seven-volume work in progress titled *The Art of Computer Programming.* Before discussing two combinatorial problems that appear for the first time in his third volume (published in January by Addison-Wesley), we consider a well-known probability paradox with a startling generalization that Knuth discovered a few years ago.

George Gamow and Marvin Stern introduced the elevator paradox in the prologue to their little book *Puzzle-Math* (Viking, 1958). Gamow once had an office on the second floor of a seven-story building in San Diego and Stern had an office on the sixth floor. When Gamow wanted to go up to see Stern, he noticed that in about five cases out of six the first elevator to stop on his floor was going down. It seemed as if elevators were being manufactured on the roof and then sent down the shafts to be stored in the basement. For Stern the situation was the opposite. When he wanted to go down to see Gamow, about five times out of six the first elevator to arrive was on its way up. Were elevators being fabricated in the basement and then sent to the roof to be carried off by helicopters?

The explanation, as Knuth pointed out later, requires a few idealizing assumptions. Suppose each elevator travels independently in continuous cycles from bottom floor to top and back again, moving with constant speed and with the same average waiting time on each floor. Thus at the time a button is pushed on any floor we can assume that each elevator is at a random point in its cycle.

For a single elevator, calculating the probability that it is on its way down when it stops on a given floor is quite easy. Stern, on the sixth floor, has five floors below and one above; therefore the probability is 5/6 that the elevator is below him and will be moving up when it arrives. Gamow, on the second floor, has five floors above and one below; therefore the probability is 5/6 that the elevator is above him and will reach his floor on its way down. Gamow and Stern explained this in their book, but then they made a slip. If there is more than one elevator, they wrote, the probabilities "of course remain the same." The slip is understandable because the statement seems so intuitively true. Apparently Knuth was the first to realize that it is not true at all! Indeed, as the number of elevators approaches infinity the probability that the first elevator to stop on any floor (except the top or bottom floors) is going up (or down) approaches exactly 1/2—a rather unexpected result. Yet the probability (for, say, the second floor) remains 5/6 for every individual elevator, and all elevators are equally likely to be the next to arrive.

The solution for two or more elevators is complicated by conditional probabilities. As Knuth puts it: "The choice of which elevator is first to arrive on the second floor is partly contingent on whether it was above us or below, since an elevator that is below the second floor when we begin to wait is likely to arrive ahead of an elevator that is above (all other things being equal)." In his 1969 paper [*see "Bibliography," page 124*] Knuth analyzes Gamow's situation as follows: Consider the portion of an elevator's route that starts at the fourth floor, then goes down to the first floor and up to the second, a total of 4/12 = 1/3 of the entire route. During the first half of this portion the elevator stops next at the second floor going down, and during the other half it will next stop going up. Therefore we may call it the unbiased portion, since it is not biased toward up or down.

If there are n elevators, Knuth now distinguishes two cases:

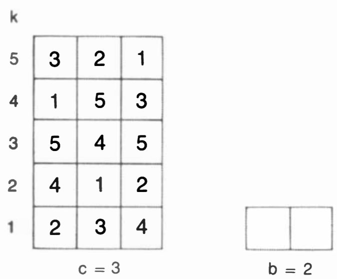
1. No elevator is in the unbiased portion. The probability of this is $(2/3)^n$, since it is 2/3 for each elevator. The next elevator to stop on the second floor will be going down.

2. At least one elevator is in the unbiased portion. The probability is $1 - (2/3)^n$. We can ignore any elevator outside the unbiased portion, since one of those in the unbiased portion will necessarily reach the second floor first. In this case the elevator will be going down with probability 1/2.

Combining these results gives a probability of $(2/3)^n + \frac{1}{2}(1 - (2/3)^n) = \frac{1}{2} + \frac{1}{2}(2/3)^n$ that the first elevator to arrive on the second floor will be going down. If there are just two elevators running in Gamow's seven-story building, the first elevator to stop at the second floor will be headed downward with probability $\frac{1}{2} + 2/9 = 13/18$. This is slightly less than 5/6, so that Gamow's chances of catching an up elevator have improved. If there are seven elevators, the probability of an elevator's going down would be 2,315/4,374, which is not far from 1/2.

Knuth gives the general formula for any building by defining p as the distance from a given floor to the bottom divided by the distance between top and bottom floors. For Gamow p is 1/6, for Stern p is 5/6. The general formula for all values of p between 0 and 1 is

$$\tfrac{1}{2} + \tfrac{1}{2}(1 - 2p)\,|1 - 2p|^{n-1}.$$



u_k/b	d_k/b	$\lceil u_k/b \rceil$	$\lceil d_k/b \rceil$
0	3/2	0	2
3/2	5/2	2	3
5/2	3/2	3	2
3/2	3/2	2	2
3/2	0	2	0
		9 +	9 = 18

Five-story-building elevator problem

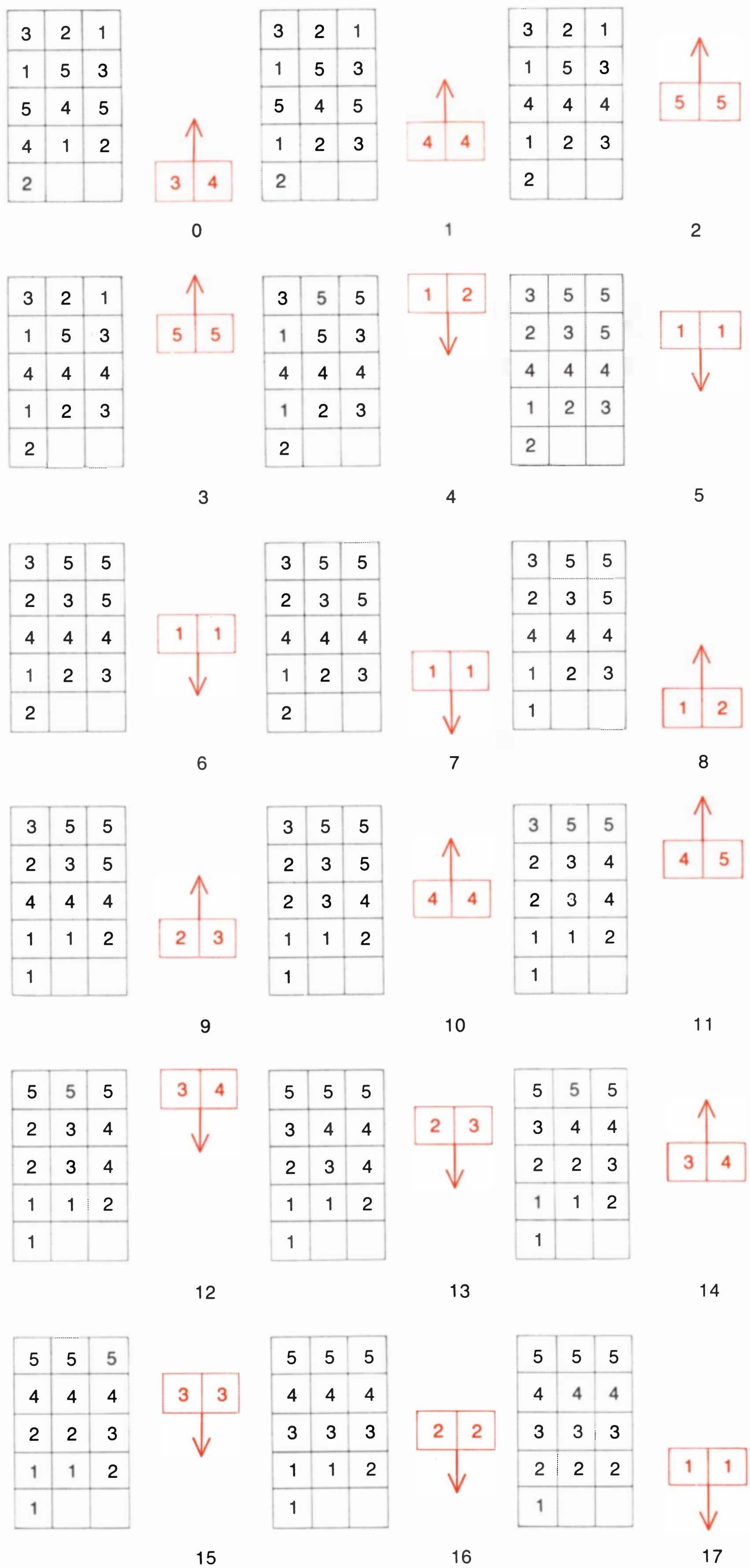
The pair of vertical lines indicate the absolute value of the expression between them. The probability approaches 1/2 as n, the number of elevators, approaches infinity.

Our second elevator problem is from the third volume of Knuth's series, a book that deals entirely with computer techniques of sorting and searching for information. Like its two predecessors, it is comprehensive in scope, written in a clear, informal style (although at times it is necessarily terse and technical) and rich in humor, historical data and problems of great recreational interest. On pages 11 through 72, for instance, Knuth brilliantly summarizes almost everything known about the combinatorial properties of permutations, a topic that ties in with scores of classic puzzle problems. The book's exercises concern such entertaining topics as solitaire card games, shuffling, anagrams, snowplows, the design of tennis tournaments (including Lewis Carroll's flawed efforts to find a design that does the best possible justice to the second-best player), rook problems, sorting puzzles, the unsolved weight-ranking problem, the Josephus problem, parking problems, Fibonacci numbers, the odd "tableaux" of Alfred Young (which have a curious relevance to the eightfold way of particle theory) and a hundred other things that lead straight into recreational mathematics.

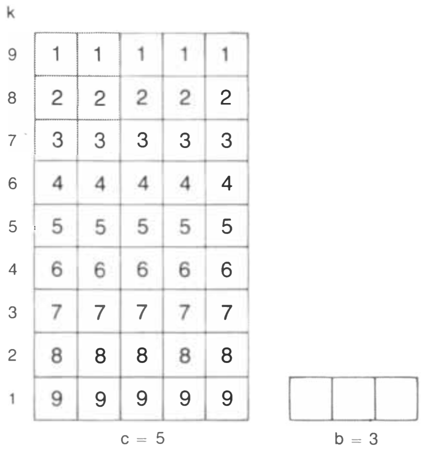
Here we are concerned with pages 357 through 360, where Knuth regards the elevator as a model of one-tape computer sorting. A building has n floors, each holding exactly c people. There is a single elevator that carries at most b people. We assume that the building is full (contains cn people). Exactly c persons want to go to each floor: c to the first floor, c to the second floor and so on. Some people may already be on their desired floor, but it is more interesting to assume that all or most are misfits who want to be on another floor.

The elevator always starts at the bottom. It moves up and down, loading and unloading passengers, until each person is where he wants to be. The elevator then returns to the first floor. A movement of the elevator from any floor to the next floor above or below will be called a unit trip. The problem is to find an algorithm that will sort all the people in a minimum number of unit trips. This operation is equivalent, of course, to minimizing the distance traveled or (assuming a constant elevator speed) to minimizing the time required for the sorting.

As Knuth points out, the people correspond to records that are to be com-



Richard M. Karp's algorithm



Nine-story-building elevator problem

puter-sorted. The building is the tape, the floors are blocks on the tape and the elevator is the computer memory. A computer can do such things as duplicate records or chop them into parts to be stored temporarily in different blocks. It turns out, however, that a clever algorithm discovered by Richard M. Karp enables the elevator to do its job with peak efficiency without having to duplicate or partition any passenger.

Let k be the number of the floor, u_k the number of misfits on k and all lower floors who want to go higher than k, and d_k the number of misfits on k and all higher floors who want to go lower than k. It is not hard to see that $u_k = d_{k+1}$. For example, suppose k equals 3. The theorem states that all people on floors 3, 2 and 1 who want to go above Floor 3 are equal in number to those on Floor 4 and higher who want to go below Floor 3. (It is like the old wine-and-water problem. In a filled building the people who go up from a bottom portion of the building must be replaced by the same number of people in the top portion who want to go down.) Both u_n (misfits on the top floor) and d_1 (misfits on the first floor) are, of course, zero, since no one wants to go above the top floor or below the first floor.

Because the elevator holds at most b people it must make at least $\lceil u_k/b \rceil$ trips from Floor k to the next floor above, where $\lceil\,\rceil$ symbolizes the roundup function (the value is rounded up to the nearest integer). Similarly, the elevator must make at least $\lceil d_k/b \rceil$ trips from k down to the next floor below. If we now calculate $\lceil u_k/b \rceil$ and $\lceil d_k/b \rceil$ for each floor, the sum of all these integers will be the least number of trips that the elevator must make to sort everyone.

Karp's algorithm achieves this minimum if u_k is not zero for any floor except the top one and provided that the number of people each floor can hold is not less than the number the elevator can hold. The procedure calls for the assumption that the elevator is always in either the UP state or the DOWN state. It starts in the UP state and repeats the following algorithm until everyone is sorted:

1. When the elevator is in the UP state, if anyone (in the elevator or on the floor where it has just stopped) wants to go up, fill the elevator with those of the highest destination, with all others remaining on the floor, then move the elevator up one floor. Otherwise change to the DOWN state.

2. When the elevator is in the DOWN state, fill it with those people of the lowest destination (who are on the elevator or on the current floor) and move the elevator down one floor. Then change the elevator to the UP state if there are no misfits on lower floors who want to go to the new current floor or higher.

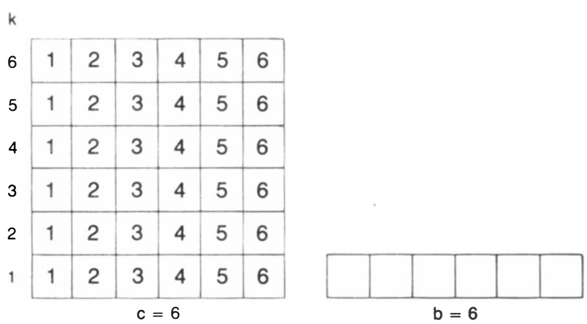
To see exactly how this operates, consider a five-floor problem [*see illustration on page 106*]. Each floor holds three people. Each person is represented by a numeral that indicates the floor he wishes to go to. The empty elevator on the right can hold only two people. In this problem all the people in the building are misfits except for one 2-person who is already on Floor 2. In order to calculate the minimum distance the elevator must travel first list the u_k/b and d_k/b values for each floor and then list these values rounded up [*see illustration*]. Note the positions of the zeros and the fact that the sequence of values for u_k/b is repeated in the d_k/b column, except that the sequence starts one floor higher. This repetition is true of all such charts and is a consequence of the theorem $u_k = d_{k+1}$. The sum of the rounded-up values is 18, so that we know the elevator must make at least 18 unit trips to accomplish the sorting and then return to the first floor.

The illustration on the preceding page shows what happens when we apply Karp's algorithm. (The final step is not shown.) Observe that occasionally people are taken off the floor on which they wish to stay. In some cases the procedure will carry a person in one direction when he wants to go in the other. "This represents," Knuth writes, "their sacrifice to the common good."

To get a feeling for the spooky way Karp's algorithm does its job, readers are urged to work out the problem of sorting 45 people in a nine-floor building with an elevator that holds three people [*see top illustration on this page*]. First calculate the minimum number of unit steps needed. Then draw the building on a sheet of cardboard, fill the rooms with small cardboard counters bearing the proper numerals and see how easy it is to apply Karp's algorithm to achieve the minimum. Of course, you can make up endless similar tasks, altering the variables k, c and b as you please and shuffling the people any way you like in the building.

If one or more floors have $u_k = 0$ (that is, no one on that floor or below wants to go above that floor), yet some higher floor has u_k greater than zero, the building becomes divided into disconnected regions. The minimum is achieved by handling each region separately according to Karp's algorithm, then piecing together the individual schedules. This procedure increases the number of unit trips by twice the number of floors that must be passed even though they have $u_k = 0$. A little experimenting on buildings with one or more $u_k = 0$ floors below the top one will make clear why that is so. It amounts to the fact that the elevator has to make special trips upward to take care of all higher disconnected regions and then return to the bottom.

Our third elevator problem, discussed on pages 374 through 376 in Knuth's third volume, is based on results obtained by Robert W. Floyd while he was working on efficient ways to rearrange



Robert W. Floyd's elevator problem

records in a magnetic-disk file. This time instead of minimizing distance we want to minimize the number of stops required by the elevator to complete the sorting. Floyd was able to establish a nontrivial lower bound, but no general algorithm is known that achieves the best possible results, except of course a brute-force trial of all possible elevator schedules.

Consider a building where the number of floors, the number of people to a floor and the elevator capacity are each six [*see bottom illustration on opposite page*]. One of Knuth's exercises is to sort the 36 people correctly by starting and ending the elevator on the first floor and to do it in no more than 12 stops. I shall give Floyd's solution next month.

Floyd's method of computing the lower bound is too complicated to explain here, but for this problem it gives nine stops. Even in this simple case it is not known whether there is a solution in nine, 10 or 11 stops. If any reader sends in a solution that is better than 12 I shall report it in a later column.

Our final problem is from Kobon Fujimura's latest Japanese puzzle book, *Dialogue about Puzzles* (Tokyo, 1971; there is no English translation), which he coauthored with Michio Matsuda. Chapter 3 is devoted to an elevator problem that is a cleverly disguised form of a well-known problem in coding theory. In a building of k floors there are n elevators. Each stops on the top and bottom floors and on exactly m floors in between (always stopping on the same m floors). We wish to determine the minimum number of elevators that will enable a person to go from any floor to any other without changing elevators. For example, suppose a building has eight floors and each elevator stops on top and bottom floors and three floors in between. One schedule for a minimum of six elevators that makes it possible for a person to go directly from any floor to any other floor is shown in the top illustration on this page.

As an introduction to this class of problems, readers are asked to answer the following question. Each elevator in a 10-floor building stops on top and bottom floors and four floors in between. What is the minimum number of elevators that will enable a person to ride from any floor to any other without changing elevators? Fujimura's solution will be given next month.

It is impossible to report on all the mechanical puzzles that are appearing on the market, but a new set of five "Piet Hein Puzzlers" calls for comment, not only because readers of this department are familiar with Piet Hein's poems and inventions but also because the new puzzles are of unusual mathematical interest. They are handsomely designed by Hubley Toys, a division of Gabriel Industries in Lancaster, Pa. I shall describe each briefly:

1. Nimbi. This is a 12-counter version of Piet Hein's nim-type game (introduced here in February, 1958) formerly called Tac Tix. The counters are locked-in, sliding pegs on a reversible circular board so that after a game is played by pushing the pegs down and turning the board over it is set for another game. Little is known so far about the best strategy for the polygonal playing field.

2. Anagog. Here we have a spherical cousin of Piet Hein's Soma cube. Six pieces of joined unit spheres are to be formed into a 20-sphere tetrahedron or two 10-sphere tetrahedrons or other solid and flat figures.

3. Crux. A solid cross of six projecting arms is so designed that each arm rotates separately. One of several problems is to bring three spots of different colors together at each intersection.

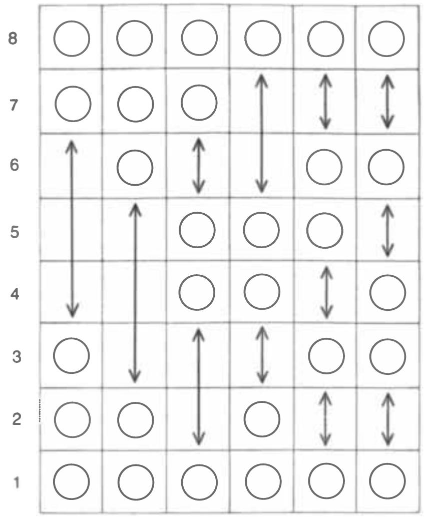
4. Twitchit. A dodecahedron has rotating faces and the problem is to turn them until three different symbols are together at each corner. The solution is unique.

5. Bloxbox. W. W. Rouse Ball, discussing the standard 14–15 sliding-block puzzle in his *Mathematical Recreations and Essays*, wrote in 1892: "We can conceive also of a similar cubical puzzle, but we could not work it practically except by sections." Now, 81 years later, Piet Hein has found an ingenious practical solution. Seven identical unit cubes are inside a transparent plastic order-2 cube. When the cube is tilted properly, gravity slides a cube (with a pleasant click) into the hole. Each cube has three black and three white sides. Problems include forming an order-2 cube (minus one corner) with all sides one color, or all sides checkered, or all striped, and so on.

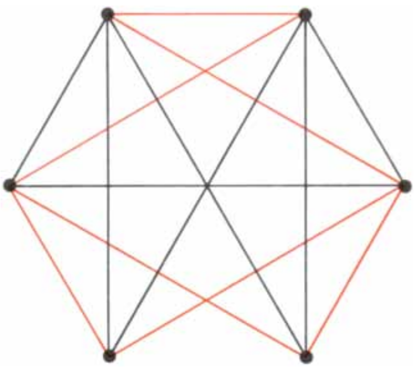
Does the parity principle involved in flat versions apply to the three-dimensional version? And what are the minimum required moves to get from one pattern to another? Bloxbox opens a Pandora's box of questions.

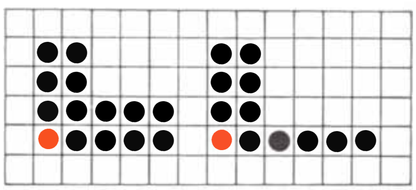
The answer to last month's Sim problem has only one basic position (variants are topologically identical) that allows the game to go 14 moves without a monochrome triangle [*see upper illustration at right*]. The 4-by-5 and 4-by-6 fields for Chomp are won by the unique first moves [*see lower illustration below*].

David Gale, who invented Chomp, has considered the game on infinite rectangular arrays. Readers may enjoy proving (on the basis of last month's theorems) that the first player wins on n-by-infinity fields (provided that n is not 2) and on infinity-by-infinity squares but loses on 2-by-infinity arrays. A new 500-page edition of *The Guide to Simulations/Games for Education and Training* mentioned last month has been published and is available from Information Resources, Inc., P.O. Box 417, Lexington, Mass. 02173.



A Japanese elevator problem


Sim game that ends on move 15


Winning chomps