MATHEMATICAL GAMES

Cram, crosscram and quadraphage: new games having elusive winning strategies

by Martin Gardner

There are many simple two-person games, for example nim, for which perfect-play strategies are known. Other games, such as ticktacktoe and dots-and-squares, may appear just as simple but actually are so complex that no strategy has yet been found except when the game is played on special fields. In spite of recent progress in computer speeds and sophisticated programming, no one yet knows whether ticktacktoe on a 4-by-4-by-4 cubical matrix can always be won by the first or second player, or whether the game is a draw if played rationally. No one knows who has the win on a dots-and-squares field of six dots on the side.

This month we consider several elegant new games that have extremely simple rules and about which relatively little is known. Some may not have general strategies; if they do, perhaps a reader of this department will be the first to discover them.

Our first game, as far as I am aware, has not been described in print, although a few mathematicians have been involved with it for at least 20 years. I originally heard of it from Geoffrey Mott-Smith, the author of several books on games and puzzles who died in 1960. He told me it had been invented by a friend who called it "plugg." Since then I have received letters from a number of mathematicians who independently invented the same game. In 1966 John Horton Conway gave it some thought, and although he did not succeed in cracking it, he did formulate a partial strategy with which the final stages of a game could be analyzed by standard nim theory.

The game can be played in various ways, all isomorphic. If the "board" is a rectangular lattice of dots in unit square formation, the rules are as follows. Players alternately draw a line that connects two orthogonally adjacent dots. No line may touch a dot after it has been joined to another. In the standard form of the game the last player to connect two dots is the winner. (In *misère,* or reverse, play the last to move is the loser.) Let us call the game "dots-and-pairs." Clearly it is a graph-theory game.

If a supply of counters is handy, one can arrange the counters to form the lattice, then each move consists in removing two orthogonally adjacent counters. Another way to play dots-and-pairs is to sketch a checkered field on paper, or outline it on graph paper. A move consists in coloring two orthogonally adjacent cells or, more simply, drawing a line that eliminates the "domino."

Still another way to play the game—the most pleasant—is to place dominoes on a checkerboard. The markings on the dominoes are of course irrelevant. All that matters is that each domino must cover just two squares. Players alternately place one domino until no further play is possible. We shall call this version of the game "cram." It is the simplest nontrivial polyomino-placing game. (Readers will be interested to know that Solomon W. Golomb's polyomino-placing game, using the 12 pentominoes, has finally reached the marketplace. Issued by the Springbok Division of Hallmark Cards Inc., this masterpiece of packaging includes the 8-by-8 board, the pieces and the authorized instructions for both the game and various puzzles. The trade name is Pentominoes. It appears just 20 years after Golomb introduced polyominoes to a group of mathematicians in his memorable talk to the Harvard Mathematics Club.)

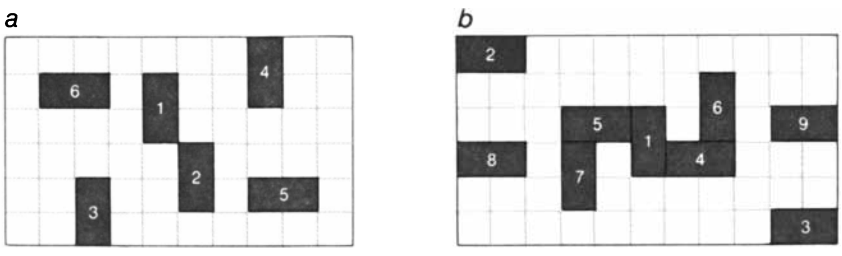
Winning strategies for cram are known for certain boards. If, for instance, the field is rectangular with two even sides and the game is standard (the last to play wins), the second player wins easily by symmetry play. He simply makes each move symmetrically opposite his opponent's last move [*see "a" in illustration on this page*]. To eliminate this strategy we can add a new rule: The second player's first move must not be symmetrical with respect to the first player's opening move. With this proviso the game can be played on a standard chessboard with 32 dominoes. It is not known which player has the win.

If standard cram is played on an even-by-odd rectangle, the first player wins by taking the two central cells, then playing symmetrically thereafter [*see "b" in illustration on this page*]. This strategy can be eliminated by denying the center to the first player on his opening move.

No general strategy is known for the reverse form of cram on even-even or even-odd fields, and no general strategy is known for standard or reverse play on odd-odd fields. Even when one of the odd sides in an odd-odd field is reduced to 1, the game is complex and still unsolved. In 1973 David Singmaster, then at the Istituto Matematico in Pisa, wrote a computer program for the 1-by-m field, standard game, with m less than 1,000. Assuming that the first player loses (because he cannot move) when m equals 0 or 1, Singmaster found 151 values of m that give the win to the second player. For m less than 100 the values are 0, 1, 5, 9, 15, 21, 25, 29, 35, 39, 43, 55, 59, 63, 73, 77, 89, 93, 97.

When m is even, the first player wins, of course, by taking the center and playing symmetrically. When m is odd, he wins for all values less than 100 that are not in the above set. I know of no computer analysis of the reverse game for 1-by-m fields.

Square fields of only the lowest orders have been investigated. The 3-by-3 game



Symmetrical winning strategy for cram on even-even fields (a) *and even-odd fields* (b) is trivial. It takes just a few minutes of analysis to see that the second player wins the standard game and loses the reverse game. The 4-by-4, with symmetry play denied the second player in standard play, takes considerably more work. Conway found it to be a second-player win in both standard and reverse play.

What about the 5-by-5? Because it is odd-by-odd, symmetry strategy is ruled out and no special rules are needed. Who has the win in standard play? In *misère*? As far as I know, neither question has yet been answered.

Cram is an "impartial" game. This means that any possible move can be made by either player. "Partial" games, or what Conway prefers to call "unimpartial" games, are those in which moves open to one player are denied to the other. Chess and checkers, for example, are partial games because each player can move only the pieces of his color. We can convert cram to a partial game by a rule proposed by Göran Andersson, who wrote to me about it in 1973.

The rule is delightfully simple. One player must make all his moves horizontally, the other must make all his moves vertically. Call it "crosscram." This, of course, instantly eliminates symmetry play from all rectangular boards. The 3-by-3, as before, can be quickly disposed of. The first player wins the standard game provided that his first move does not include a corner cell. The second player wins the reverse game.

Crosscram in the 4-by-4 form is sufficiently complicated to make a good pencil-paper game [*see top illustration on this page*]. I am too lazy to analyze it but would welcome hearing from readers who do, and I shall report on results in a later column.

Both cram and crosscram can be regarded as special cases of more general games. Cram is Piet Hein's Tac Tix, now more commonly called nimbi. (See Chapter 15 of *The Scientific American Book of Mathematical Puzzles & Diversions*.) Nimbi is usually played with a square array of counters. A move consists in removing as many orthogonally adjacent counters as desired, not just a pair, from any orthogonal row. Crosscram is a special case of partial nimbi in which the rules restrict one player to horizontal rows and the other to vertical.

Another interesting form of partial nimbi was invented in 1972 by James Bynum of Tacoma, Wash., who has permitted me to describe it here. It is the same as partial nimbi except that each play must have a maximum length; that is, the orthogonally adjacent cells must be bounded on each end by either the field's border or an opponent's perpendicular move. The game's first move must necessarily be an entire row or column [*see second illustration from top at right*].

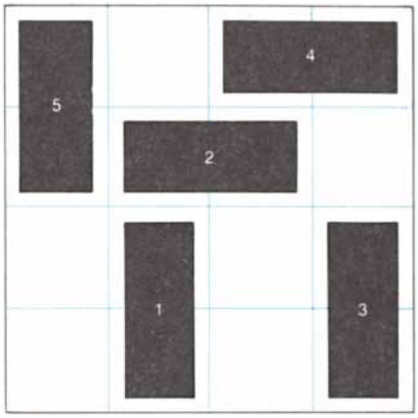
Bynum's game was solved in 1973 by Conway. *Misère* play is almost trivial. The second player wins on all square boards, and if the board is a nonsquare rectangle, the player whose moves parallel the shorter dimension wins regardless of whether he goes first or second. The winning rule is: Pick one of the two sides that parallel your moves, then always play as close to that side as possible. Standard play is more interesting. The first player has the win on all square boards as well as on all nonsquare rectangles with sides of the same parity (even-even or odd-odd). On even-odd rectangles the player whose moves parallel the even dimension wins regardless of who plays first.

Interested readers may enjoy seeing if they can develop a set of strategic rules that will ensure a win for the player who has the win. The game yielded readily to Conway's unpublished theory of unimpartial games. I shall say no more about his analysis because it will be included in a book on unimpartial game theory that Conway is reported to be writing.

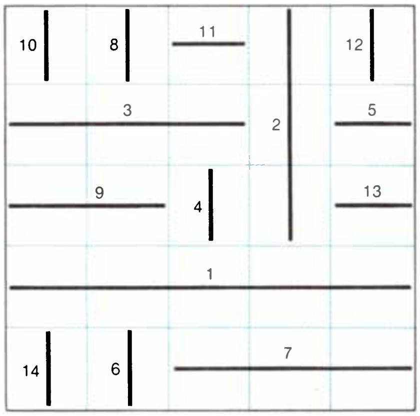
Quadraphage (square-eater) is a partly explored family of games invented and named by David L. Silverman in the late 1940's. He suggested the basic idea to Richard A. Epstein, who mentions one version briefly on page 406 of his *Theory of Gambling and Statistical Logic* (Academic Press, 1967). Silverman discusses two other versions on page 186 in his book of game puzzles, *Your Move* (McGraw-Hill, 1971). Elwyn Berlekamp has done considerable work on quadraphage, which he will summarize in a book on games that he is preparing in collaboration with Conway and Richard K. Guy. Here I shall introduce only some of the game's simpler aspects.

Quadraphage games are played on a chessboard of side n, usually square. Pieces consist of one chess piece, usually a king, and a supply of counters. The counters are the quadraphages, which I shall call quads for short. Each quad "eats" the cell on which it is placed, thus preventing the king from moving to that cell. The game starts with the king on the central square if the board is odd-odd, or on one of the four central squares if the board is even-even. The rest of the board is empty. One player moves the king in the usual manner. The other player places counters, q at a time, on Q cells. As in the game of go, counters do not move once they are placed. The object of the king is to get safely to the edge of the board. The quads try to box in the king so that it cannot escape. The quad player conventionally goes first, then the players move alternately.

If q equals 4 (four squares eaten on



Standard crosscram with a first-player win



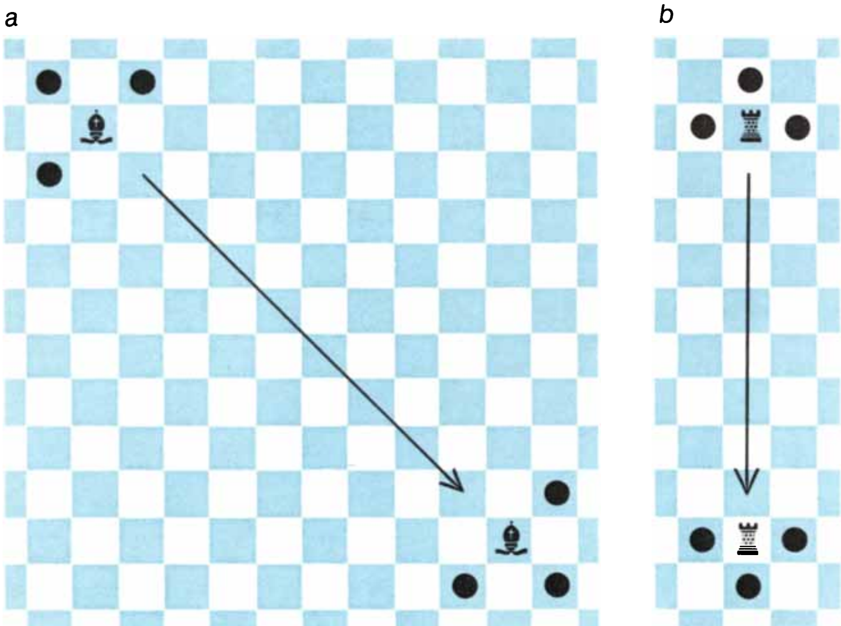
Bynum's game with a second-player win

Two quads to trap king on an 8-by-8 board each move), it is easy to show that the king can be captured in no more than three moves on all boards of side 5 or greater. (Of course, the king escapes immediately on a 4-board.) If q equals 3, it takes only a little more effort to find that the king can be trapped on boards of side 6 or greater.

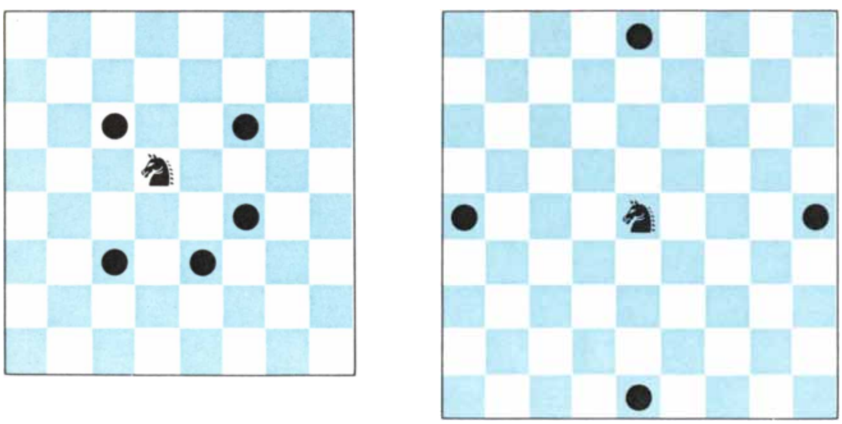
When q equals 2, the game starts to get interesting. Although I have not proved it, I believe the king can escape on the 7-board but can be trapped on the standard 8-board and all higher boards. The strategy is to move first as shown in the bottom illustration on the preceding page, then continue by placing quads two at a time on only the white border cells except when the king is next to the border and the quads must be placed side by side to prevent its escape.

What about $q = 1$? Can the king always escape no matter how large the board? Surprisingly, it cannot. Berlekamp has proved that on a board as small as 33-by-33 the king is lost. Unfortunately both proof and strategy are too complicated to give here. Golomb has shown that if the king's moves are limited to orthogonal moves and q equals 1, the king escapes on the 7-board but can be trapped on the 8-board.

Although the king escapes easily on a standard chessboard, a pleasant game (proposed by Silverman) results if the king tries to maximize its moves before reaching the border. Quads, placed one per move, try either to trap the king or to force it to the border as quickly as possible. If the king escapes, it is awarded one point for each quad on the board. No points are awarded if the king is trapped. The game is particularly enjoyable when it is played on the order-18 go board.



How three quads per move trap bishop (a) *and rook* (b) *on infinite boards*



Trapping a knight on order-8 board (left) *and order-9 board* (right)

The go board, with its large supply of stones, is a handy tool for working on the many unsolved quadraphage problems. What, for instance, happens when the king is replaced by a different chess piece? If the piece is a bishop, rook or queen, we must limit the length of its move to avoid triviality. Assume that the board is infinite but the piece cannot move a distance of more than, say, a billion squares. With this limitation a bishop is easily captured in a goose chase along a diagonal by placing three quads per move as shown in *a* in the upper illustration on this page. Once the ends are blocked the bishop can be confined to the diagonal. Three quads per move can similarly trap a rook in a goose chase along an orthogonal, as shown in *b*, and seven can trap a queen along either an orthogonal or a diagonal. Can the bishop or rook be trapped by two quads per move? By one? Can the queen be trapped by fewer than seven? I shall give answers to some of these questions next month.

If the piece is a knight, we must consider it free when it lands on a cell that is second from the border because on its next move it can leap over a border cell. On boards of sides 5, 6 and 7 the centered knight has eight moves to freedom, so that eight quads obviously are needed to trap it. Five quads per move will trap the knight on the order-8 board, and four per move are sufficient for the order-9. The lower illustration on this page shows one of several winning first moves for each board, although I must add that I am not certain all this information is correct.

Can three quads per move trap a knight on an infinite board? One quad per move surely is insufficient, although Berlekamp reports he has yet to see a rigorous proof.

Correction: The chart in last December's column, which listed simplest-known expressions by cubes for integers 1 through 99, had two errors. In the sum for 51 the sign for 602^3 should be positive and in the sum for 79 the sign for 33^3 should be negative.