MATHEMATICAL GAMES

Games of strategy for two players: star nim, meander, dodgem and rex

by Martin Gardner

Consider a two-person game with the following characteristics: (1) It is a game of perfect information, that is, both players have complete knowledge of the game's structure after every move. (2) The players move alternately. (3) Decisions are not made by chance. (4) The game ends after a finite number of moves with a win by one player. (No draw is possible.)

It is not hard to see that there must be a winning strategy for either the first player or the second. If the first player (henceforth called *A*) does not have a winning strategy, he must lose. This means that the second player, *B*, has a winning strategy. Does the argument apply if we rescind the requirement that the game end in a finite number of moves?

Curiously, it all depends on whether or not one accepts the "axiom of choice." This notorious axiom says that from any collection (finite or infinite) of nonempty sets, with no elements in common, you can form a new set by taking one element from each set. In the 1930's Stefan Banach, Stanislaw Mazur and Stanislaw Ulam discovered a type of infinite game in which neither *A* nor *B* has a winning strategy if the axiom of choice is accepted. Someone argued that this proves the Unitarian dogma that there is "at the most" one God, because if two gods could play such a game, neither could know a winning strategy and therefore neither could be called omniscient!

That, however, is by the way. Here we shall examine some new two-person nonchance games for which the rules are extremely simple and for which a winning strategy is either known or capable of being known. All but one of the games are played with counters on boards that are easily drawn on cardboard. Two differently colored sets of counters, such as go stones or small poker chips, will be useful for any reader who wants to play or to analyze the games.

An example of an almost trivial game of the nim type, but one with a strategy that is not immediately apparent, is played on the star pattern shown in the illustration below. Put a counter on each of the star's nine points. *A* and *B* take turns removing either one counter or any two counters joined by a straight line segment. The player who takes the last counter wins.

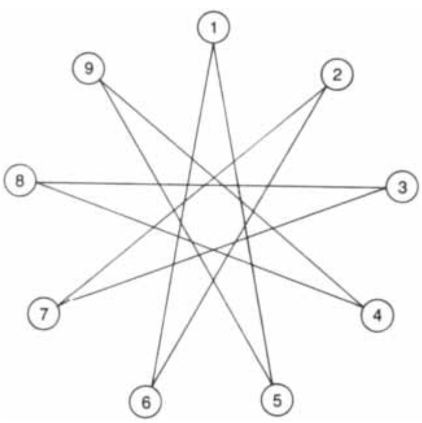
B can always win at star nim by a strategy based on the board's symmetry. Imagine that the black lines are strings. The pattern can be opened up to a circle that is topologically equivalent to the star. If *A* takes one counter from this circle, *B* takes the two counters that are directly opposite. If *A* takes two counters, *B* takes one counter that is directly opposite. In each case two sets of three counters are left. Now, whatever *A* takes from one set, *B* takes the corresponding counter or counters from the other set. Obviously *B* will get the last counter. If the reader plays a few games on the circle, translating each move to its equivalent on the star, he will soon see how to use the star's symmetry for playing the strategy.

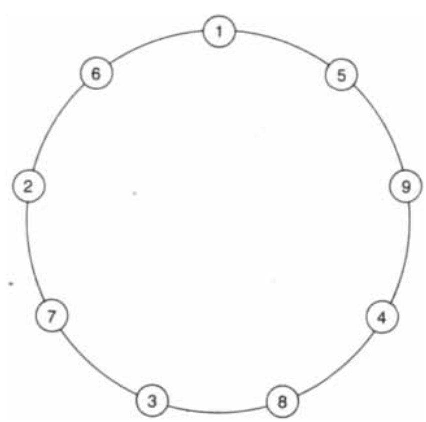
In the late 1960's G. W. Lewthwaite of Thurso in Scotland invented a delightful game with an artfully concealed "pairing strategy" that gives the second player a sure win. On a 5-by-5 square matrix place 13 black counters and 12 white counters in alternating checkerboard fashion. Any one of the black counters, say the one in the center, is removed [*see top illustration at left on opposite page*].

Player *A* controls the white counters and *B* the black. They take turns moving one of their counters orthogonally to the vacant square until a player loses by being unable to move. If the board is colored like a checkerboard, it is obvious that on each move a counter goes to a square of different color and that no counter can be moved twice. The game therefore cannot go beyond 12 moves for each player. It may end before then, however, in a win for either player unless *B* plays rationally.

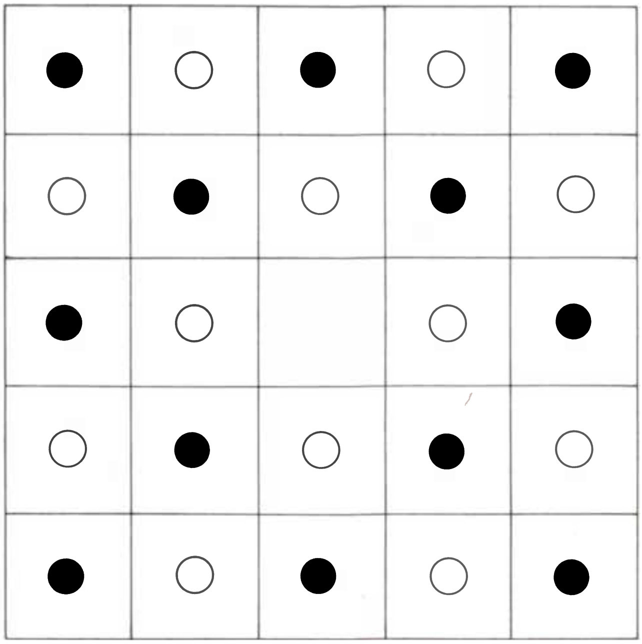
B's strategy is to imagine that the matrix, except for the initially vacant cell, is covered with 12 nonoverlapping dominoes. It does not matter how they are placed. The top illustration at the right on the opposite page shows a sample pattern. Whenever *A* moves, *B* simply moves his counter that is on the domino *A* has just vacated. Since this ensures that *B* always has a move to follow a move by *A*, *B* is sure to win in 12 or fewer moves.

The game can be played not only with counters but also with square tiles or cubes that slide within a matrix surrounded by a rim. Suppose the rules are amended to allow either player at any time to move a row or a column of three counters (or tiles), provided that the two end counters are of his color. This is a splendid example of how an apparently trivial alteration of a rule can enormously complicate a game's analysis. Lewthwaite was unable to find a winning strategy for either player in this variant of his game.

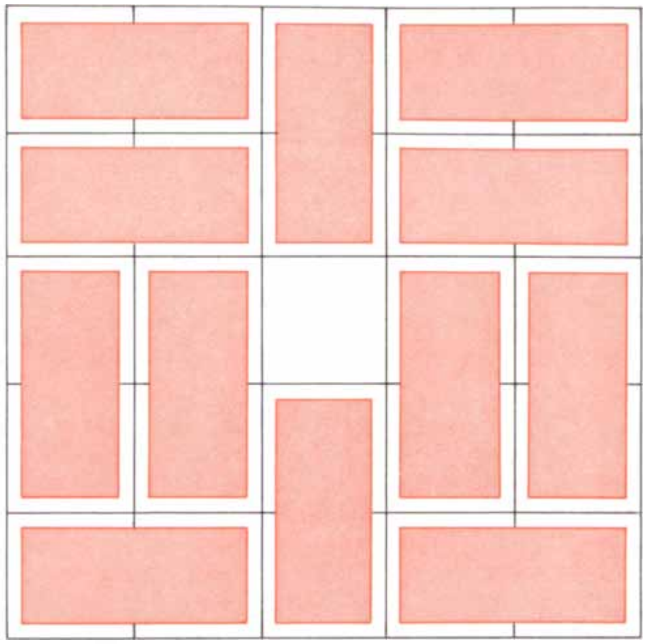




***Star nim* (left) *and its winning strategy* (right)**



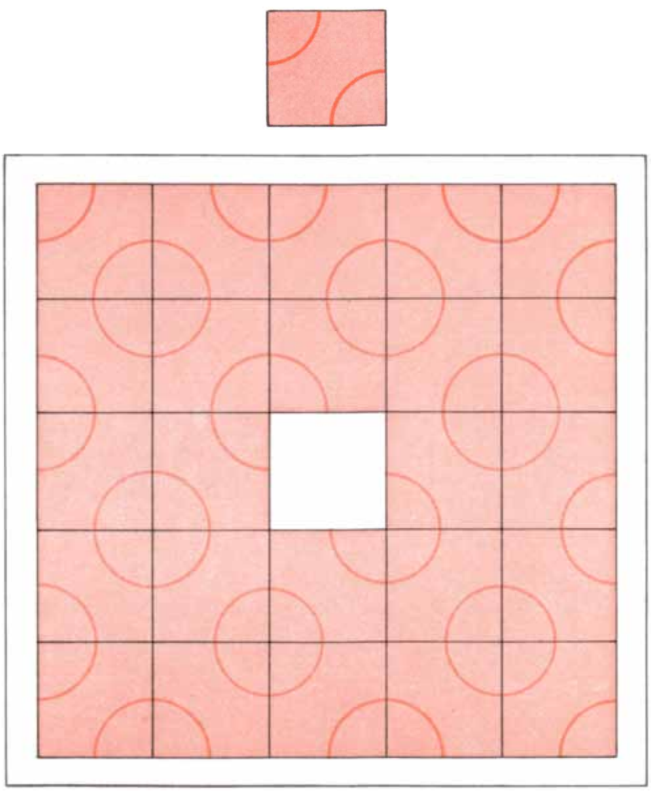
G. W. Lewthwaite's counter game



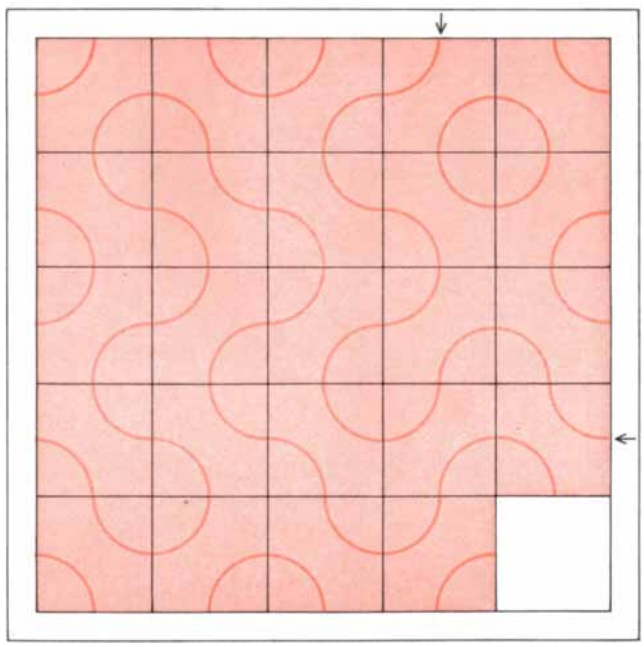
A pairing strategy for Lewthwaite's game

Games based on the sliding of unit squares within a square matrix offer a plethora of unexplored possibilities. Lewthwaite proposes an attractive game that he calls meander. It uses 24 identical tiles placed in a 5-by-5 tray to form the pattern shown in the illustration at the left below. The players take turns sliding a single counter or a straight row or column of two, three or four counters. The play continues until a player wins by creating a pattern in which at least three tiles form a continuous line or path that joins two edges (opposite or adjacent) of the tray. The illustration at the right below shows a winning pattern. The game is probably too complex for solving without a computer program, and perhaps too complex for solving even with one.

In 1972, when Colin Vout was a mathematics student at the University of Cambridge, he invented an intriguing counter game that he calls dodgem be-



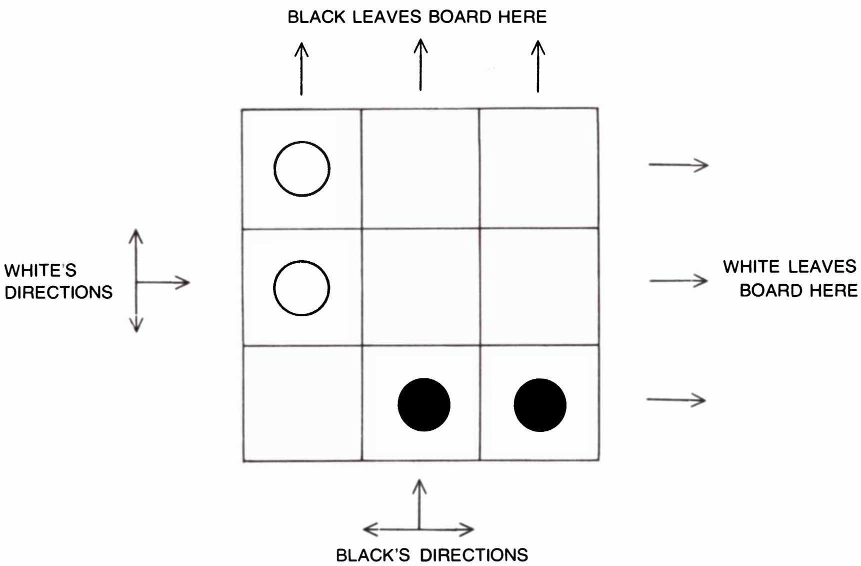
Meander, with example of pattern on a tile at top



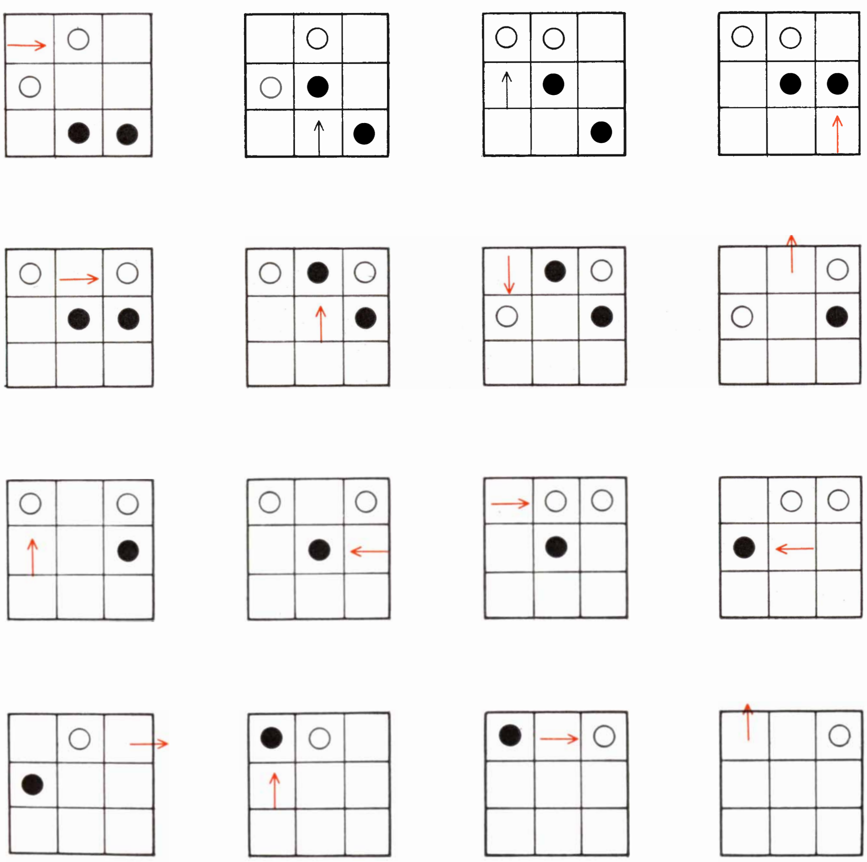
A possible winning pattern in meander

cause it is so often necessary for a piece to dodge around enemy pieces. It is playable on a checkerboard of any size. Even the game on a 3-by-3 board is complicated enough to be interesting.

Two black counters and two white ones are initially placed as shown in the upper illustration below. Black sits on the south side of the board and White sits on the west. The players alternately move a counter one cell forward or to their left or right, unless it is blocked by another counter of either color or by an edge of the board. Each player's goal is to move all his pieces off the far side of the board. In other words, Black moves orthogonally north, west or east and attempts to move both of his pieces off the north side of the board. White moves east, north or south and tries to move his pieces off the east side of the board.



Colin Vout's dodgem



A dodgem game won by Black

There are no captures. A player must always leave his opponent a legal move or else forfeit the game. The first to get all his pieces off the board wins. The lower illustration on this page shows a typical game won by Black.

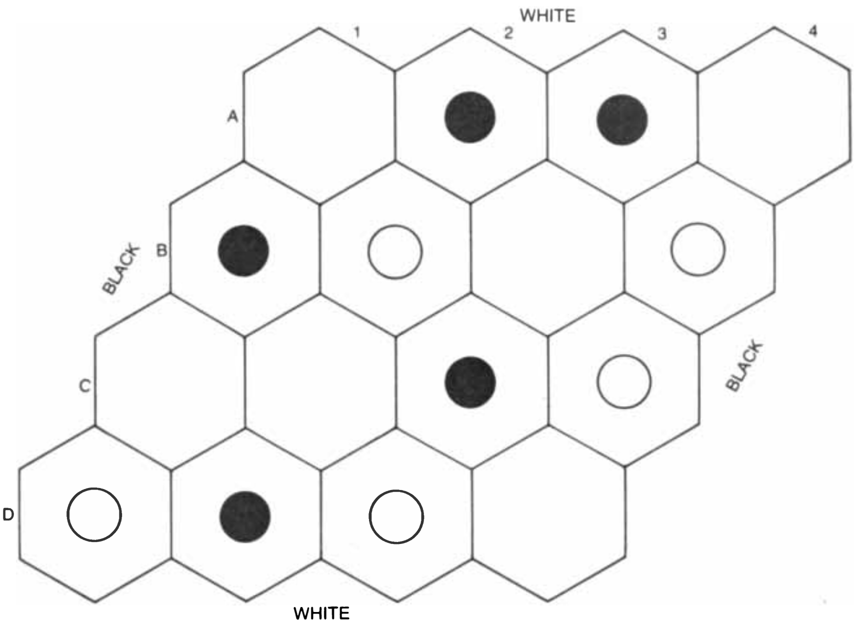
Vout assures me that the first player has the win on the order-3 board, but as far as I know no games on higher-order boards have yet been solved. On a board of side n each player has $n - 1$ pieces placed on the west and south edges, with the southwest corner cell vacant. Played with seven checkers or pawns of one color and seven of another color on the standard order-8 checkerboard or chessboard, it is a most enjoyable game.

Piet Hein's now classic game of hex (see Chapter 8 of *The Scientific American Book of Mathematical Puzzles & Diversions*) remains unsolved except for small boards. For readers unfamiliar with the game, it is played on an n-by-n rhombus of hexagons such as the order-4 board shown in the top illustration on the opposite page. White opens by placing a white counter on a cell. Black follows with a black counter. They take turns placing counters on vacant cells (there are no moves or captures) until a player wins by forming a chain of adjacent counters that joins his side of the board to the opposite side, White by joining the north and south edges, Black by joining the east and west edges.

It is easy to see that no draw is possible. There is a famous proof by John F. Nash (who independently invented hex) that on a rhombus of any size the first player has a winning strategy, although the proof gives no hint of what the strategy is. Suppose White allows Black to tell him where he must make his first move. Can White still always win if he plays rationally? This modified version of hex has been called Beck's hex after Anatole Beck, who both proposed and solved it. Writing on hex in Chapter 5 of *Excursions into Mathematics*, by Beck, Michael Bleicher and Donald W. Crowe (Worth Publishers, 1969), Beck proves that if White opens in an acute corner cell, Black has a winning strategy if he plays next to it, on his first row. As Bleicher comments in a footnote, this "wrecks Beck's hex."

What about *misère*, or reverse, hex, known as rex, in which the first player to join his sides loses? As is so often the case in two-person games, the reverse game proves to be much harder to crack. No general strategy is known, although Robert O. Winder, in unpublished arguments, has shown the existence of a first-player winning strategy in rex of even order and a second-player winning strategy for all odd orders. More recently Ronald J. Evans has carried Winder's arguments a step further by showing that on even-order boards there is a winning strategy if White opens in the acute corner (see Evans' paper "A Winning Opening in Reverse Hex," in *Journal of Recreational Mathematics,* Volume 7, Summer, 1974, pages 189–192).

Rex on the order-2 rhombus is trivial, and it is not difficult to analyze exhaustively on the order-3 rhombus. Play on the order-4 rhombus is so complicated, however, that even though it is known that an acute-corner opening initiates a win, the strategy itself remains unformulated. The position shown in the top illustration on this page is an order-4 rex problem composed by Evans. Can the reader determine White's only correct move? It will be given next month.



Rex, a reverse hex game, with White to play and win

Here is an even simpler game for which no general strategy is known. It is played on a 1-by-n board (a single row of n squares) with counters that are all alike. A and B take turns placing a counter until one player wins by getting three counters adjacent. Could anything be simpler? A can always win when n is odd by first taking the center cell, then playing symmetrically opposite his opponent thereafter. For even n, however, things are not so simple. On most even rows A seems to have the win, but not necessarily, and the exceptions follow no known rule. Take $n = 6$, for example. The reader may enjoy working it out to see who has the win.

John Horton Conway has pointed out that this game is equivalent to a game I called 1-by-n cram in this department for February, 1974, except that it is played with trominoes instead of dominoes. It is easy to see the isomorphism. In playing the game as described above it is obviously disastrous to place a counter either next to another counter or one cell from it, since either move gives the opponent an instant win. Hence we might as well prohibit both moves. An easy way to do it is to require that each play consist of a triplet of adjacent counters, which is the same as placing a tromino on the field. (The middle of the triplet corresponds to placing a single counter, and the ends of the triplet enforce the two new rules.) The winner is the player who places a tromino last. (To complete the equivalence we must allow the placing of a tromino at either end of the field so that it extends one cell beyond the end.) Of course, the game can also be played by forming a row of n counters and removing them by alternate moves of taking three adjacent counters.

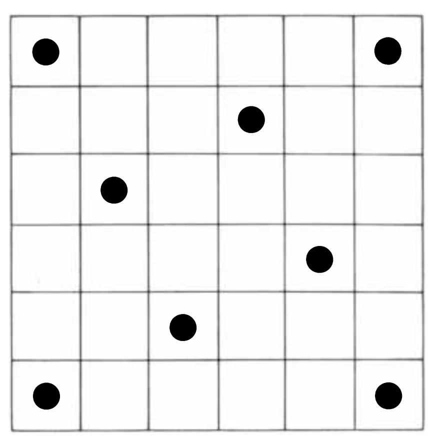
This triplet version of cram is considered in a classic paper by Richard K. Guy and Cedric A. B. Smith, "The G-Values of Various Games" (*Proceedings of the Cambridge Philosophical Society,* Volume 52, July, 1956, pages 514–526), where it is coded as Game .007. Elwyn Berlekamp has computer-analyzed the game to very high n without finding any periodicity in the Grundy numbers, which means that no one is even close yet to a general rule. The *misère* version of 1-by-n tromino cram, like its domino counterpart, is also unsolved.

Ulam has proposed extending the counter form of tromino cram to a square matrix. The players take turns placing single counters until one player wins by getting three in a row orthogonally or diagonally. As before, odd-order fields are trivial because the first player wins by taking the center and then playing symmetrically until his opponent offers a win. On even-order boards the order 4 is trivial, but no one yet knows who has the win on orders 6 or 8. The bottom illustration on this page, supplied by Ulam, shows a position on the order-6 board for which the next player must lose.

Here again we can play an equivalent game by alternately placing polyominoes, in this case squares of nine cells, but it is not very convenient because in addition to allowing the pieces to extend into a unit border around the field we must also allow them to overlap one another by just two cells (corner and side). No one has even begun to find a general strategy for the game in standard or reverse form.

Last month's problem concerned nine girls. Each day for 12 days a triplet of girls goes out for a walk. How can the triplets be formed so that each pair of girls appears in exactly one triplet?

H. S. M. Coxeter, in the much revised 12th edition of W. W. Rouse Ball's *Mathematical Recreations and Essays* (University of Toronto Press, 1974), shows how the unique solution (known



Stanislaw Ulam's triplet game

as a Steiner triple system of order 9) is obtained from the following matrix:

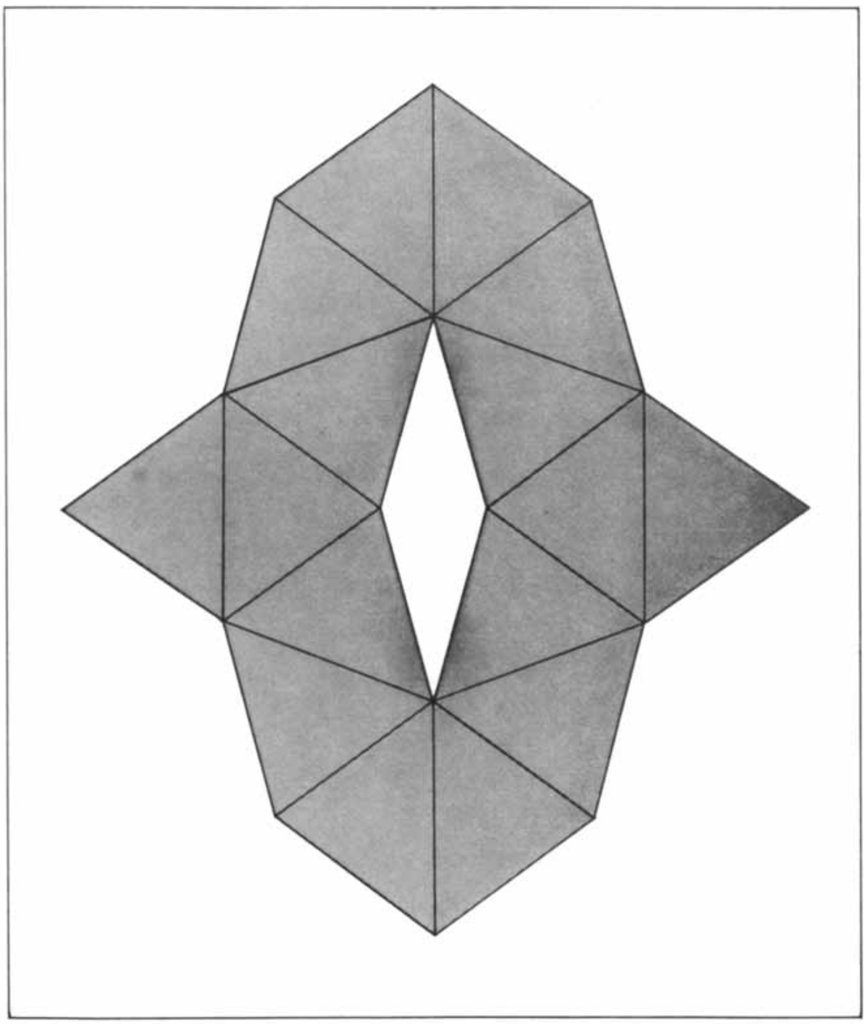
a b c
d e f
g h i

The rows (*abc, def, ghi*) provide three triplets. The columns (*adg, beh, cfi*) provide three more. The main diagonals (*aei, ceg*) and the broken diagonals (*bfg, cdh, afh, bdi*) complete the list of 12 triplets.

The orders of Steiner triple systems are numbers that have a remainder of 1 or 3 when they are divided by 6. The next higher system, order 13, has two basic solutions. Order 15 is known to have 80 solutions.

A model of Kurt Schmucker's one-hole toroid with 48 congruent equilateral triangle faces, mentioned last month, is easy to make. It consists of a ring of eight regular octahedrons joined by their faces [*see illustration below*]. Schmucker found that rings could be made by joining eight replicas of each of the Platonic solids except the tetrahedron. No matter how many tetrahedrons are joined by their faces, no ring is possible even when the solids are allowed to intersect one another. A proof is given by J. H. Mason in his paper "Can Regular Tetrahedra Be Glued Together Face to Face to Form a Ring?" in *The Mathematical Gazette,* Volume 56, October, 1972, pages 194–197.

So many letters were received on the short problems in the March issue that I can report on only a few of them. Scores of readers solved the problem of the worm, some using the discrete model, others assuming continuous stretching of the rubber rope. It is assumed, naturally, that the worm is an ideal worm of point size and eternal life span. Some notion of the enormous length of the rope when the worm finally reaches the end can be gained from an observation by H. E. Rorschach. If the rope starts with a cross-sectional area of one square kilometer, it ends up as a single line of atoms, the space between each adjacent pair of atoms being many times the size of the known universe. The time lapse is comparably greater than the age of the universe. (The length of the rope was erroneously given in centimeters. It should have been kilometers.)



Ring of eight octahedrons

For playing the integer-choice game I suggested a spinner for randomizing selections from the set 1, 2, 3, 4, 5. Walter Stromquist proposed using a pair of dice as follows:

"After a few beers you cannot be expected to distinguish the fives and sixes, so that if either of these numbers appears on either die, you will have to roll again. Also, since you are really using only 2/3 of each die, it is only natural to multiply the total by 2/3 (dropping all fractions) before writing it down. For example, the largest number you can roll with two dice (without rolling again) is 8, so that the largest number you would ever choose is 2/3 of that, or 5. Out of 16 plays you should expect to choose with these frequencies: 1 once, 2 five times, 3 four times, 4 five times, 5 once." These are precisely the desired frequencies for playing the best strategy.

C. Stanley Ogilvy wrote to say he had included the three-circle problem in his book *Excursions in Geometry*. He too assumed that the proof by way of the three cones took care of all cases, but one day, after giving the problem to his class at Hamilton College, a student pointed out that the proof does not apply when a small sphere is between two larger ones. In such cases it is not possible for the two intersecting planes to be mutually tangent to all three spheres.

Many readers sent in other ways of proving the theorem. Richard I. Felver, Clyde E. Holvenstot, David B. Shaw and Radu Vero each proposed turning the drawing so that the line (on which the intersections of the tangent pairs lie) is horizontal and above the circles. The circles can now be viewed as being equal spheres inside three mutually intersecting pipes of identical circular cross section, seen in perspective. The tangent lines become the parallel sides of the three pipes. Since the pipes all must rest on a plane, their parallel sides seen in perspective will all have vanishing points on the horizon line.

It is not necessary that the circles be nonintersecting; indeed, the theorem can be stated in a more general way, in terms of "centers of similitude" instead of tangents, to hold for circles that lie entirely within one another. I am indebted to Donald Keeler for explaining this, as well as pointing out that the theorem is known as Monge's theorem after the French mathematician and friend of Napoleon, Gaspard Monge, who described it in a 1798 treatise. R. C. Archibald, writing in *American Mathematical Monthly*, Volume 22, 1915, page 65, traced the theorem (says Keeler) back to the ancient Greeks.

Daniel Sleator found that the theorem has an analogue with four spheres in space. Each of the four triplets will have the apexes of its three cones on a straight line. Because these four lines intersect one another at six points the four lines must be coplanar. Therefore the vertexes of the six cones all lie on a plane. Perhaps the theorem generalizes to all higher spaces.

The chess problem, to my surprise, has a second basic solution that was found by so many readers I cannot list all their names. Black opens with P-Q4, then follows with either P-KN3, P-KR4 or N-KB3. (Black's first two moves can be interchanged.) On his third move he checks with Q-Q3, then mates with Q-KB5.

The discussion of D. R. Kaprekar's self-numbers contained three errors: 101 is generated by 100 (not 101), 100,000 is generated by 99,959 (not 99,950) and the years of this century that are self-numbers begin with 1906 and 1917.

Donald R. Woods of Princeton University was the first to report on a computer program for John Harris' cube-rolling problem. It produced an unexpected result. The minimum-move solution is 36, two fewer than the one given last month. Moreover, aside from its reversal and rotations and reflections, it is unique:

URD LLD RRU LDL URD RUL
DLU URD RUL DRD LUL DRU

This was confirmed by the computer programs of John L. Couch, Robert Harris, Ken Jackman, John MacDonald, David Nixon, Jerome J. Sjostrand, John Sweeney and David Vanderschel. I shall report later if any readers found the 36 solution by hand or by other computer programs before its publication here. Harris writes that he has lowered the number of moves for his second problem (mentioned in April) from 84 to 74.

So many letters have been received on the six "April Fool" hoaxes that appeared in this department in April that it has been impossible to reply to any of them. In the next issue, however, I hope to comment in a general way on how the hoaxes were received, and to make some observations on the jokes themselves.