MATHEMATICAL GAMES

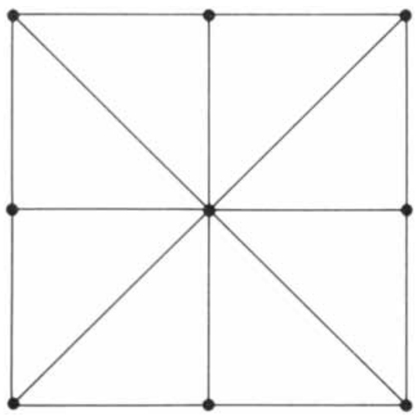
In which players of ticktacktoe are taught to hunt bigger game

by Martin Gardner

The world's simplest, oldest and most popular pencil-and-paper game is still ticktacktoe, and combinatorial mathematicians, often with the aid of computers, continue to explore unusual variations and generalizations of it. In one variant that goes back to ancient times the two players are each given three counters, and they take turns first placing them on the three-by-three board and then moving them from cell to cell until one player gets his three counters in a row. (I discussed this game in my first column on ticktacktoe, in March, 1957.) Moving-counter ticktacktoe is the basis for a number of modern commercial games, such as John Scarne's Teeko and a new game called Touché, in which concealed magnets cause counters to flip over and become opponent pieces.

Standard ticktacktoe can obviously be generalized to larger fields. For example, the old Japanese game of go-moku ("five stones") is essentially five-in-a-row ticktacktoe played on a go board. Another way to generalize the game is to play it on "boards" of three or more dimensions. These variants and others were discussed in this department for August, 1971.

In March, 1977, Frank Harary devised a delightful new way to generalize ticktacktoe. Harary is a mathematician who divides his time between the University of Michigan's mathematics department and its Institute for Social Research. He has been called Mr. Graph Theory because of his tireless, pioneering work in this rapidly growing field that is partly combinatorial and partly topological. Last year he gave a speech in the German city Xanten, thereby completing a list of about 150 cities outside the U.S. in which he has lectured that includes names beginning with the letters from *A* through *Z*. Harary is the founder and editor of *Journal of Graph Theory* and the author of *Graph Theory,* which is considered the world over to be the definitive textbook on the subject. His papers on graph theory, written alone or in collaboration with other mathematicians, number some 300. Harary ticktacktoe, as I shall call his generalization of the game, opens up numerous fascinating areas of recreational mathematics. This month, with Harary's permission, I shall summarize for the first time some of the basic results that have been obtained so far.



Ticktacktoe as a graph-coloring game

We begin by observing that standard ticktacktoe can be viewed as a two-color geometric-graph game of the type Harary calls an achievement game. Replace the nine cells of the ticktacktoe board with nine points joined by lines, as is shown in the illustration at the left. The players are each assigned a color, and they take turns coloring points on the graph. The first player to complete a straight line of three points in his color wins. This game is clearly isomorphic with standard ticktacktoe, which is well known to end in a draw if both players make the best possible moves.

Let us now ask: What is the smallest square on which the first player can force a win by coloring a straight three-point path? It is easy to show that it is a square of side 4. Harary calls this side length the board number b of the game. It is closely related to the Ramsey number of generalized Ramsey graph theory, a number that plays an important part in the Ramsey games discussed in this department for November, 1977. (Ramsey theory is a field in which Harary has made and continues to make notable contributions. It was in a 1972 survey paper on Ramsey theory that Harary first proposed making a general study of games played on graphs by coloring the graph lines.) Once we have determined the value of b we can ask a second question. In how few moves can the first player win? A little doodling shows that on a board of side 4 the first player can force a win in three moves. Harary calls this figure the move number m of the game.

In ticktacktoe a player wins by taking cells that form a straight, order-3 polyomino that is either edge- or corner-connected. (The corner-connected figure corresponds to taking three cells on a diagonal.) Polyominoes of orders 1 through 5 are depicted in the illustrations on pages 23 and 25. The polyomino terminology was coined by Solomon W. Golomb, who was the first to make a detailed study of these figures. Harary prefers to follow the usage of a number of early papers on the subject and call them "animals." I shall follow that practice here.

We are now prepared to explain Harary's fortuitous generalization. Choose an animal of any order and declare its formation to be the objective of a ticktacktoelike game. As in ticktacktoe we shall play not by coloring spots on a graph but by marking cells on square matrixes with noughts and crosses in the usual manner or by coloring cells red and green as one colors edges in a Ramsey graph game. Each player tries to label or color cells that will form the desired animal. The animal will be accepted in any orientation and, if it is asymmetrical, in either of its mirror-image forms.

Our first task is to determine the animal's board number, that is, the length of the side of the smallest square on which the first player can, by playing the best possible strategy, force a win. If such a number exists, the animal is called a winner, and it will be a winner on all larger square fields. If there is no board number, the animal is called a loser. If the animal chosen as the objective of a game is a loser, the second player can always force a draw, but he can never force a win. The proof of this fact is well known and applies to most ticktacktoelike games. Assume that the second player has a winning strategy. The first player can "steal" the strategy by first making an irrelevant opening move (which can never be a liability) and thereafter playing the winning strategy. This finding contradicts the assumption that the second player has a winning strategy, and so that assumption must be false. Hence the second player can never force a win. If the animal is a winner and b is known, we next seek m, the minimum number of moves in which the game can be won.

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For the 1-cell animal (the monomino), which is trivially a winner, b and m are both equal to 1. When, as in this case, m is equal to the number of cells in the animal, Harary calls the game economical, because a player can win it without having to take any cell that is not part of the animal. The game in which the objective is the only 2-cell animal (the domino) is almost as trivial. It is also economical, with b and m both equal to 2. The games played with the two 3-cell animals (the trominoes) are slightly more difficult to analyze, but the reader can easily demonstrate that both are economical: for the L-shaped 3-cell animal b and m are both equal to 3, and for the straight 3-cell animal b equals 4 and m equals 3. This last game is identical with standard ticktacktoe except that corner-connected, or diagonal, rows of three cells are not counted as wins.

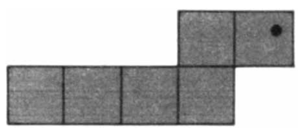
It is when we turn to the 4-cell animals (the tetrominoes) that the project really becomes interesting. Harary has given each of the five order-4 animals names, as is shown in the illustration on this page. Readers may enjoy proving that the b and m numbers given in the illustration are correct. Note that Fatty (the square tetromino) has no such numbers and so is labeled a loser. It was Andreas R. Blass, one of Harary's colleagues at Michigan, who proved that the first player cannot force Fatty on a field of any size, even on the infinite lattice. Blass's result was the first surprise of the investigation into Harary ticktacktoe. From this finding it follows at once that any larger animal containing a two-by-two square also is a loser: the second player simply plays to prevent Fatty's formation. More generally, any animal that contains a loser of a lower order is itself a loser. Harary calls a loser that contains no loser of lower order a basic loser. Fatty is the smallest basic loser.

The proof that Fatty is a loser is so simple and elegant that it can be explained quickly. Imagine the infinite plane tiled with dominoes in the manner shown at the top of the illustration on page 26. If Fatty is drawn anywhere on this tiling, it must contain a domino. Hence the second player's strategy is simply to respond to each of his opponent's moves by taking the other cell of the same domino. As a result the first player will never be able to complete a domino, and so he will never be able to complete a Fatty. If an animal is a loser on the infinite board, it is a loser on all finite boards. Therefore Fatty is everywhere a loser.

Early last year Harary and his colleagues, working with only the top four domino tilings shown in the illustration on page 26, established that all but three of the 12 5-cell animals are losers. Among the nine losers only the one containing Fatty is not a basic loser. Turning to the 35 6-cell animals, all but four contain basic losers of lower order. Of the remaining four possible winners three can be proved losers with one of the five tilings shown in the illustration. The animals that can be proved basic losers with each tiling pattern are shown alongside the pattern. In every case the proof is the same: it is impossible to draw the loser on the associated tiling pattern (which is assumed to be infinite) without including a domino; therefore the second player can always prevent the first player from forming the animal by following the strategy already described for blocking Fatty. There are a total of 12 basic losers of order 6 or lower.

It is worth noting how the tiling proof that the straight animal of five cells is a loser (another proof that was first found by Blass) bears on the game of go-moku. If the game is limited to an objective of five adjacent cells in a horizontal or vertical line (eliminating wins by diagonal lines), the second player can always force a draw. When diagonal wins are allowed, the game is believed to be a first-player win, although as far as I know that has not yet been proved even for fields larger than the go board.

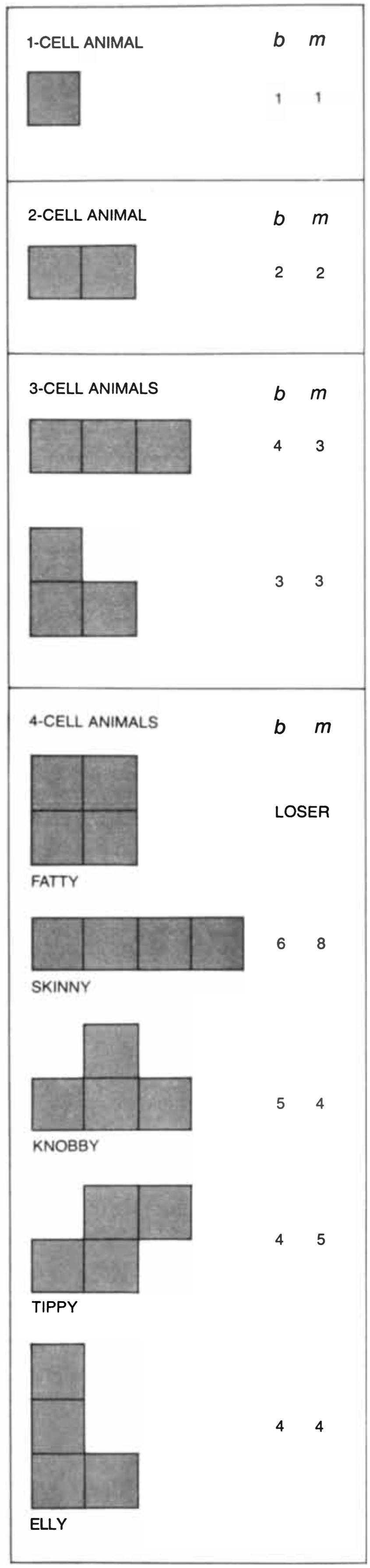
The only 6-cell animal that may be a winner is the one Harary and his colleagues call Snaky:



Although they have not yet been able to prove this animal is a winner, they believe its board number is no larger than 15 and its move number is no larger than 13. This assertion is the outstanding unsolved problem in Harary ticktacktoe theory. Perhaps a reader can prove Snaky is a loser or conversely show how the first player can force the animal on a square field and determine its board and move numbers.

All the 107 order-7 animals are known to be losers. Therefore since every higher-order animal must contain an order-7 animal, it can be said with confidence that there are no winners beyond order 6. If Snaky is a winner, as Harary and his assistant Geoffrey A. Exoo conjecture, there are exactly a dozen winners—half of them economical—and a dozen basic losers.

Any 4- or 5-cell animal can be the basis of a pleasant pencil-and-paper game. If both players know the full analysis, then depending on the animal chosen either the first player will win or the second player will force a draw. As in ticktacktoe between inexpert players, however, if this knowledge is lacking, the game can be entertaining. If the animal chosen as the objective of the game is a winner, the game is best played on a board of side b or $b - 1$. (Remember that a square of side $b - 1$ is the largest



Animals of 1 cell through 4 cells

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board on which the first player cannot force a win.)

All the variations and generalizations of Harary ticktacktoe that have been considered so far are, as Harary once put it, "Ramseyish." For example, one can play the misère, or reverse, form of any game—in Harary's terminology an avoidance game—in which a player wins by forcing his opponent to color the chosen animal. Only four animals are known to be nondraws in avoidance games. The smallest are the monomino and the domino. When the domino is the "objective," the second player can force a win on any rectangular board (including a square) for which both sides are at least 2 in length. The second player's winning strategy is to assume that the board is colored like a checkerboard and then to respond to his opponent's opening move by playing on any cell of the same color. Thereafter he continues to play anywhere he likes on cells of that color until they are all occupied. It is an interesting problem to provide a rigorous demonstration of the second player's subsequent strategy.

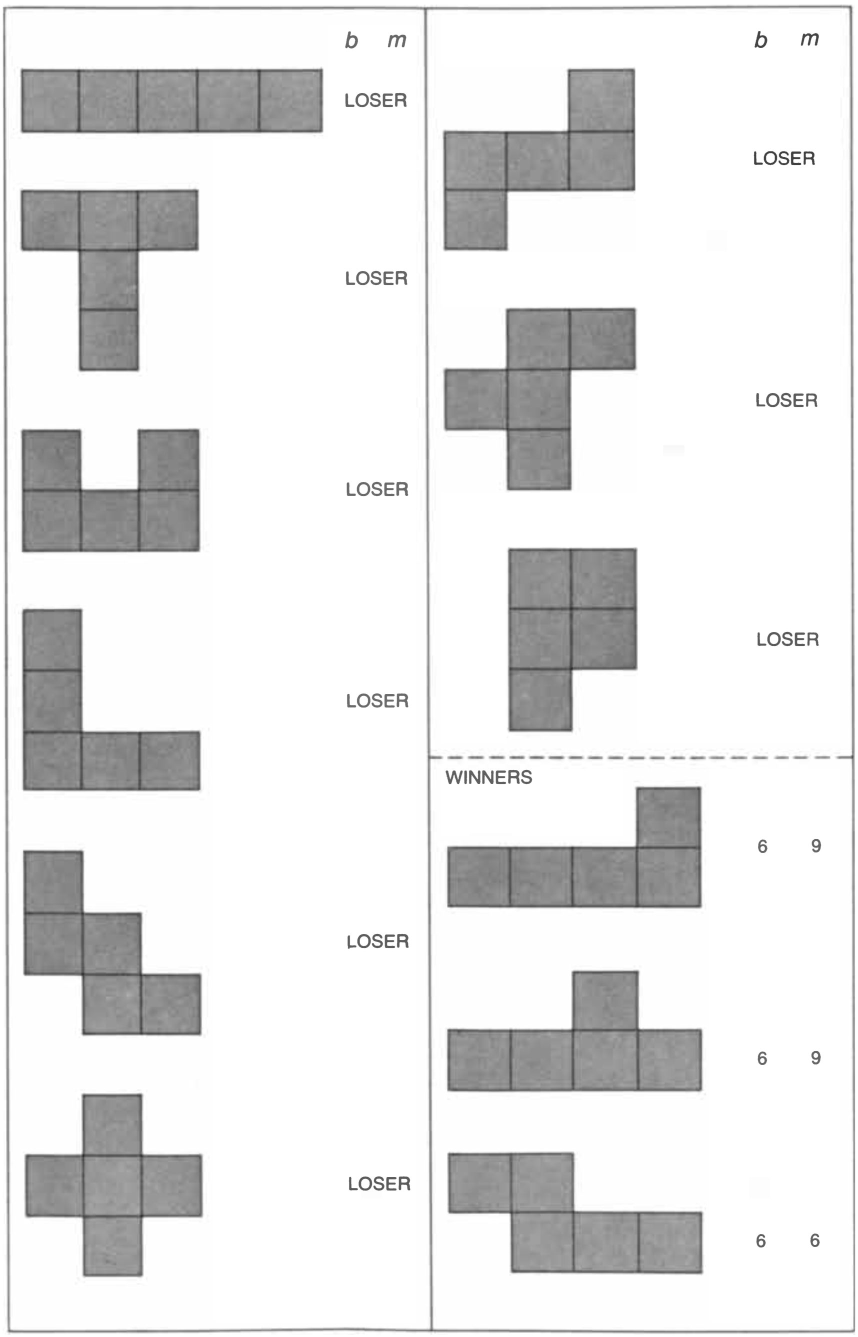
On a square board the first player can also be forced to complete the *L*-shaped 3-cell animal. Obviously the length of the square's side must be at least 3 for the game to be meaningful. If the length of the side is odd, the second player will win if he follows each of his opponent's moves by taking the cell symmetrically opposite the move with respect to the center of the board. If the first player avoids taking the center, he will be forced to take it on his last move and so will lose. If he takes it earlier in the game without losing, the second player should follow with any safe move. If the first player then takes the cell that is symmetrically opposite the second player's move with respect to the center, the second player should again make a harmless move, and so on; otherwise he should revert to his former strategy. If the length of the square's side is even, this type of symmetrical play leads to a draw, but the second player can still win by applying more complicated tactics.

On square boards the straight 3-cell animal cannot be forced on the first player. The proof of this fact is a bit difficult, even for the three-by-three square, but as a result no larger animal containing the straight 3-cell species can be forced on any square board. (The situation is analogous to that of basic losers in animal-achievement games.) Hence among the 4-cell animals only Fatty and Tippy remain as possible nondraws. Fatty can be shown to be a draw on any square board, but Tippy can be forced on the first player on all square boards of odd side. The complete analysis of all animal-avoidance games is still in the early stages and appears to present difficult problems.

Harary has proposed many other nontrivial variants of the basic animal games. For example, the objective of a game can be two or more different animals. In this case the first player can try to form one animal and the second player the other, or both players can try to form either one. In addition achievement and avoidance can be combined in the same game, and nonrectangular boards can be used. It is possible to include three or more players in any game, but this twist introduces coalition play and leads to enormous complexities. The rules can also be revised to accept corner-connected animals or animals that are both edge- and corner-connected. At the limit, of course, one could make any pattern whatsoever the objective of a ticktacktoelike game, but such broad generalizations usually lead to games that are too complicated to be interesting.

Another way of generalizing these games is to play them with polyiamonds (identical edge-joined equilateral triangles) or polyhexes (identical edge-joined regular hexagons) respectively on a regular triangular field or a regular hexagonal field. One could also investigate games played with these animals on less regular fields.

The games played with square animals can obviously be extended to

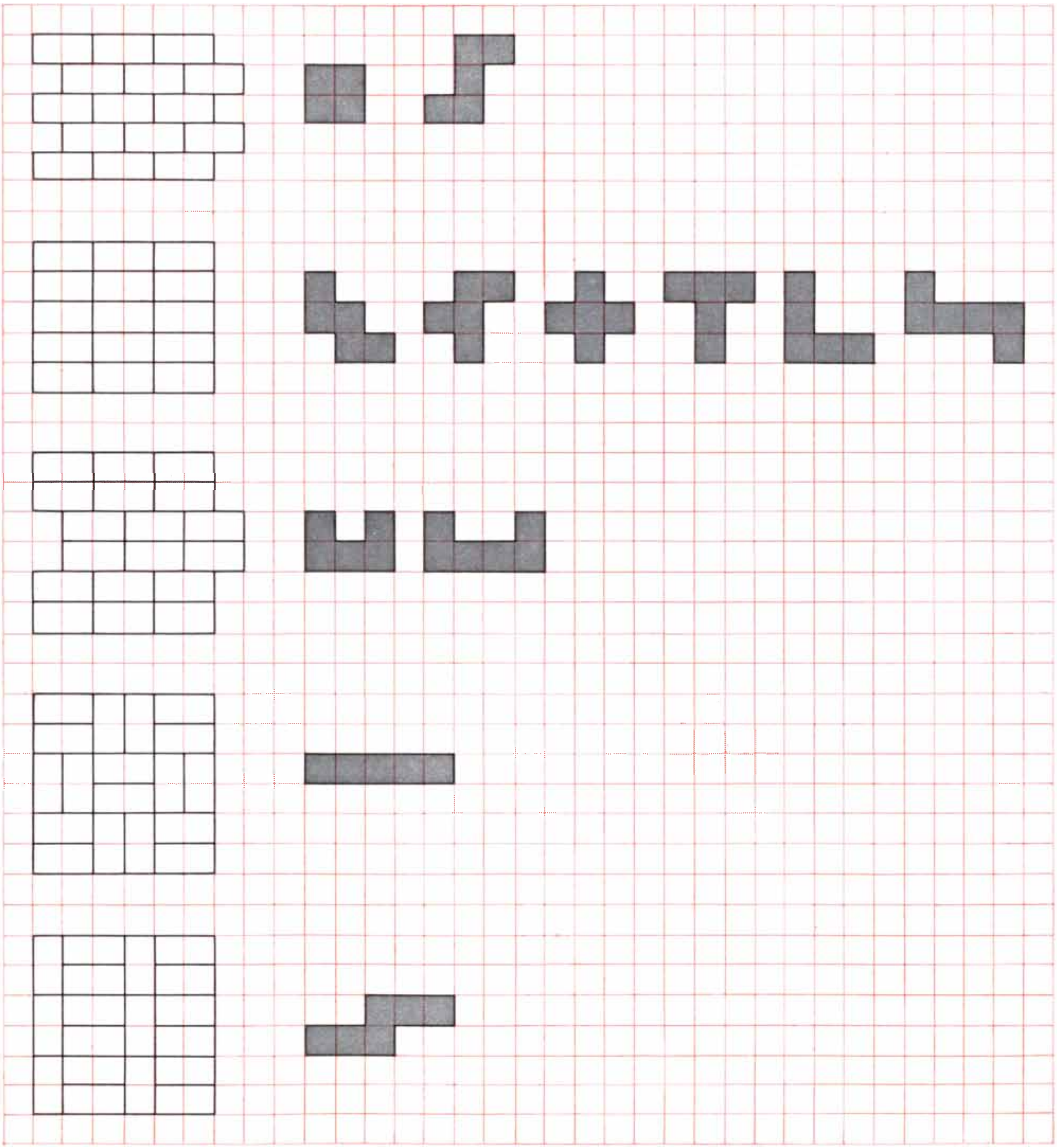


The 12 animals of 5 cells

boards of three or more dimensions. For example, the 3-space analogue of the polyomino is the polycube: *n* unit cubes joined along faces. Given a polycube, one could seek *b* and *m* numbers based on the smallest cubical lattice within which the first player can force a win and try to find all the polycubes that are basic losers. This generalization is almost totally unexplored.

As I have mentioned, Blass, now associate professor at Michigan, is one of Harary's main collaborators. The others include Exoo, Jerald A. Kabell and Heiko Harborth, who is investigating games with the triangular and hexagonal cousins of the square animals. Harary is planning a book on achievement and avoidance games in which all these generalizations of ticktacktoe and many other closely related games will be explored, and he is also developing computer programs for playing these games both offensively and defensively.

The "impossible" hypercard discussed in this department last November suggested a number of variants. Paul T. Merva and Alexis A. Gilliland came on the funniest variant independently. They each sent in a model of the paper ring shown in the illustration on page 28. It is actually a Möbius band, and in the form depicted it seems even more mysterious than the original card. Gilliland also applied the construction to the lid of a cylindrical can and to a Klein bottle. In the case of the Möbius band as many mystery flaps as are wanted can be made in what seems to be an untwisted ring. The ring will be one- or two-sided depending on whether an odd



Tiling patterns **(left)** *for the 12 basic losers* **(right)**

or an even number of flaps are made. On a cardboard napkin ring (or a ring cut from a cylindrical container) it is possible to make an even number of such flaps without having to cut the ring apart and rejoin the ends.

Scores of readers wrote to insist that the answer to the social-security-number problem given in this department for last December is not unique and to offer alternate solutions. I found the situation mystifying until I realized what had happened: they had all relied on a pocket calculator with an eight-digit readout to divide an eight-digit number by 8, without checking by hand to see if there was a remainder! (Incidentally, a large number is divisible by 8 if and only if its last three digits are.) Many other readers sent correct proofs of the uniqueness of the number 3816-54729, based either on computer programs or on familiar divisibility rules plus reasoning.

Michael R. Leuze worked with a computer to examine solutions to this problem in number systems other than base 10. He found that there are no solutions in any odd base or in base 12. In base 2 there is the trivial solution 1. In base 4 there are two solutions, 123 and 321; in base 6, 14325 and 54321; in base 8, 3254167, 5234761 and 5674321. In base 14, as in base 10, there is a unique solution: 9 12 3 10 5 4 7 6 11 8 1 2 13. Leuze conjectures that there are no solutions in higher bases.

David M. Sanger generalized in a different way by asking what the numbers are whose first n digits are divisible by n, with no other requirements. His computer program found all such numbers, starting with the 45 two-digit numbers (ignoring initial zeros) and ending with the unique 25-digit number 3,608,528,-850,368,400,786,036,725.

Because no 26th digit can be added to make a number divisible by 26 that number ends the sequence. There are 2,492 10-digit numbers; the smallest number is 1,020,005,640 and the largest one is 9,876,545,640. One peculiar number is 3,000,060,000. The number of numbers increases steadily from $n = 2$ through $n = 9$ and $n = 10$ (in both cases there are 2,492 numbers) and then declines steadily. For $n = 20$ through $n = 25$ the numbers of numbers are respectively 44, 18, 12, 6, 3 and 1. The last is the only number that provides a single answer for any n.

Alan Delahoy was the first of the readers too numerous to mention who sent proofs that 16 moves is minimal for the knight-switch problem. Many of these proofs begin by transforming the problem to a 12-point graph in the manner given in my book *Aha! Insight* (W. H. Freeman and Company, 1977), in the first printing of which I incorrectly gave 18 as the minimum number of moves. When the problem is in this form, it is easy to show that a solution must have an even number of moves that cannot be less than 14. (If the board has only three knights of one color, seven moves are needed to get them from one side to the other.) All that remains, then, is to show that 14 moves are too few. The insight that reduces 18 moves to 16 is the realization that backtracking one knight—returning it to a cell from which it had moved—clears the way for another knight move. All 16-move solutions have this feature.

Howard Rumsey, Jr., proved on a home computer that any distribution of the six knights can be reached from the starting pattern in 22 moves or fewer. Any knight pattern can be reached from any other pattern in no more than 26 moves. Rumsey found seven pairs of such patterns (not counting symmetries) for which 26 moves are required to go from one distribution of knights to the other.

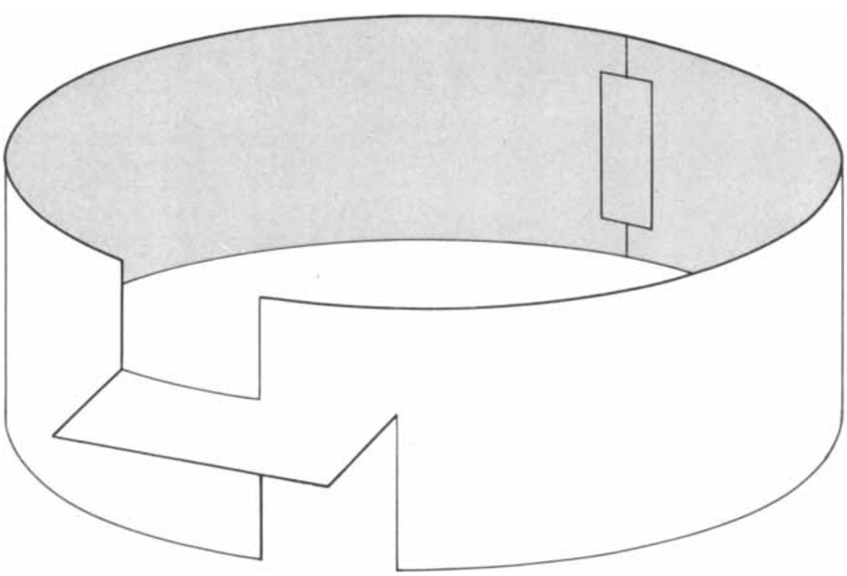
James G. Mauldon suggested replacing the middle knight on each side with a king that moves like a knight and adding the requirement that the kings change places. He found a 22-move solution to the problem. If knights are placed on the two central squares to make four knights of the same color on each side, the knights of different colors can be switched in 12 moves. Mauldon also added the proviso that these eight knights be paired with respect to the horizontal center line of the board and that all pairs be interchanged. He believes 36 moves are minimal here. He has found a 44-move solution to the problem that pairs the knights that are symmetrically opposite with respect to the center of the board.

The robot ASMOF gave "inkstand" and "crankshaft" as dictionary words containing respectively *nkst* and *nksh*. Many readers sent in other words. The commonest word for *nkst* was "prankster," but "clinkstone," "pinkster," "pinkstone," "sinkstone," "stinkstone" and archaic words found only in *The Oxford English Dictionary* were received. For *nksh* among the most common alternatives suggested were "bankshare," "monkship," "monkshood" and "tankship." Some readers composed sentences with solution words. Jim Rector closed his letter with "No thankstoyou, I thinkstoomuch."

George Starbuck found "schnappsed," another 10-letter word of one syllable, but he was topped by his friend William Harmon (both men are poets), who wrote to him: "Schnappsed can't be beat. I realized this while being broughammed to the airport." (Several dictionaries prefer the monosyllabic pronunciation of this 11-letter word.)

The robot's reasoning as to why a person cannot be exactly one-third Scottish, one-third Chinese and one-third Hungarian came in for heavy criticism. Readers were right in not taking the robot's solution seriously, since it relied on several unrealistic assumptions. In particular, ASMOF's argument assumes that all progenitors are full-blooded, which in the real world is found to be preposterous as soon as someone's ancestry is traced back a few generations.

One final correction: At the end of this department for last December a prime-number proof was incorrectly attributed to Anna M. Penk. She is the mother of Michael A. Penk, a computer scientist, whose name should have appeared instead.



Mystery flap on a Möbius band