MATHEMATICAL GAMES

Casting a net on a checkerboard and other puzzles of the forest

by Martin Gardner

No tree in all the grove but has
its charms,
Though each its hue peculiar.
—William Cowper, *The Task, Book 1: The Sofa*

In graph theory, which is the study of the structures formed by joining points with lines, a tree is a connected network of line segments that includes no circuits. A circuit is a closed path that allows one to travel along a connected network from a given point back to itself without retracing any lines. It follows that any two points on a tree are joined by a unique path. Trees are extremely important in graph theory, and they have endless applications in other branches of mathematics, particularly probability theory, operations research and artificial intelligence.

Suppose a finite set of n points are randomly scattered about in the plane. How can they be joined by a network of straight lines that has the shortest possible total length? The solution to the problem has practical applications in the construction of such networks as roads, power lines, pipelines and electrical circuits. If no new points are allowed to be added to the original set, the shortest network connecting them is called a minimal spanning tree. It is easy to see the network must be a tree: if it included a circuit, one could shorten it at once by removing a line from the circuit.

There are many ways to construct a minimal spanning tree. The simplest is known as a greedy algorithm, because at each step it bites off the most desirable piece. It was published in 1956 by Joseph B. Kruskal, now at the AT&T Bell Laboratories. First find two points that are closer together than any other two and join them. If more than one pair of points are equally close, choose any such pair. Repeat the procedure with the remaining points in such a way that joining a pair never completes a circuit. The final result is a spanning tree of minimal length.

A minimal spanning tree is not necessarily the shortest network spanning the original set of points. In most cases a shorter network can be found if one is allowed to add more points. For example, suppose you want to join the three points defining the corners of an equilateral triangle. Two sides of the triangle make up a minimal spanning tree. The spanning tree can be shortened by more than 13 percent by adding an extra point at the center and then making connections only between the center point and each corner [*see upper illustration on this page*]. Each angle at the center is 120 degrees.

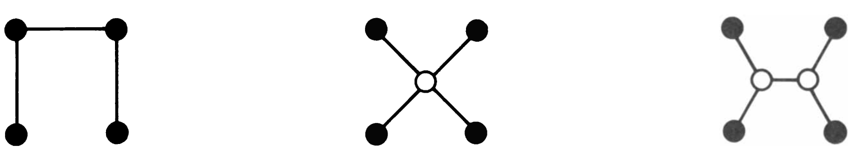
A less obvious example is the minimal network spanning the four corners of a square. You might suppose one extra point in the center would give the minimal network, but it does not. The shortest network requires two extra points [*see lower illustration on this page*]. Again all the angles around the extra points in the network are 120 degrees. The network with one extra point in the center has length $2\sqrt{2}$, or about 2.828. The network with two extra points reduces the total length to $1 + \sqrt{3}$, or about 2.732.

One of the first mathematicians to investigate such networks was Jakob Steiner, an eminent Swiss geometer who died in 1863. The extraneous points that minimize the length of the network locally are now called Steiner points. (I shall describe what is meant by "locally" below.) It has been proved that all Steiner points are junctions of three lines forming three 120-degree angles. A tree with Steiner points is called a Steiner tree. Although adding Steiner points can reduce the length of the spanning tree, a Steiner tree is not always the shortest network spanning the original set of points. When it is, it is called a minimal Steiner tree.

Minimal Steiner trees are almost always shorter than minimal spanning trees, but the reduction in length may depend on the length of the original spanning tree. It has been conjectured that for any given set of points in the plane, the length of the minimal Steiner tree cannot be less than $\sqrt{3}/2$, or about .866, times the length of the minimal spanning tree; the result has been proved, however, only for three, four and five points. Just as a set of points can have more than one minimal spanning tree, so it can have more than one minimal Steiner tree, although of course all minimal Steiner trees for a given set of points have the same length. A Steiner tree can have at most $n - 2$ Steiner points, where n is the number of points in the original set.

Many simple Steiner trees can be found empirically by a simple analog device you can build. Two parallel sheets of Plexiglas are joined by perpendicular rods that correspond to the

How a minimal spanning tree **(left)** *is shortened if an extra point is allowed* **(right)**



Minimal spanning tree of a square, shortened by adding one or two extra points

A. K. Dewdney is on vacation. His "Computer Recreations" column will appear again next month.

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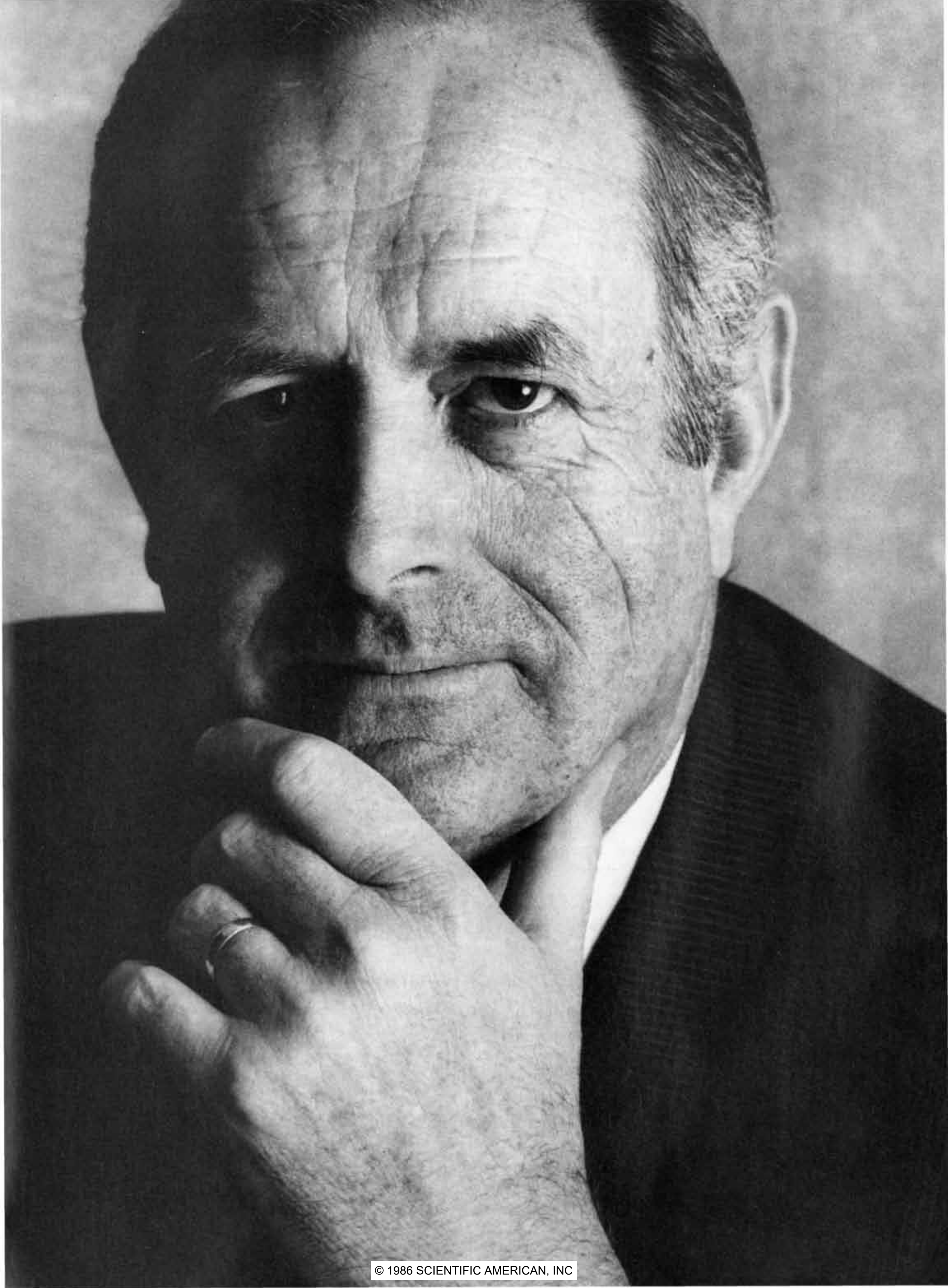
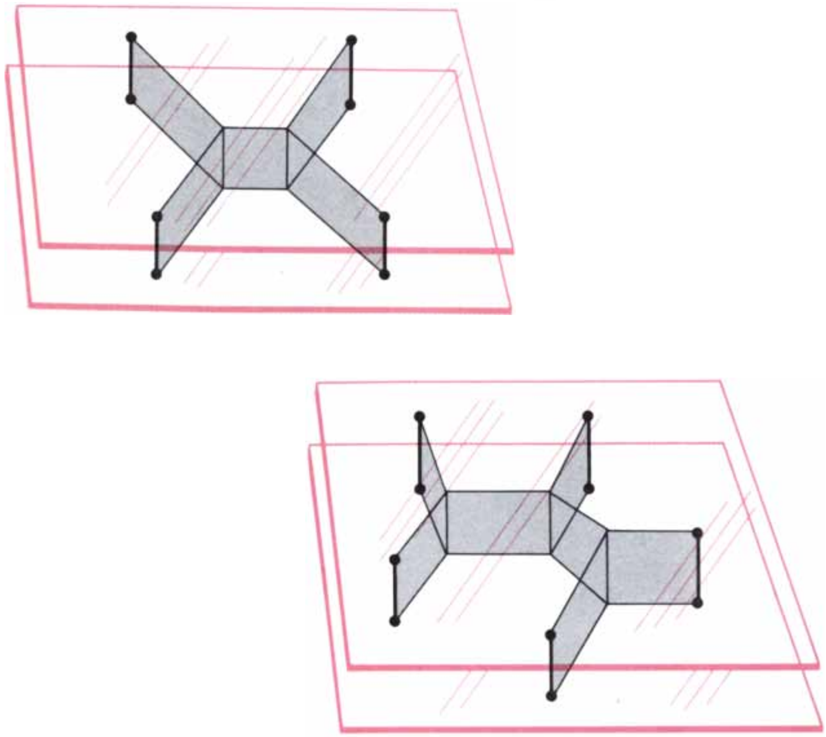
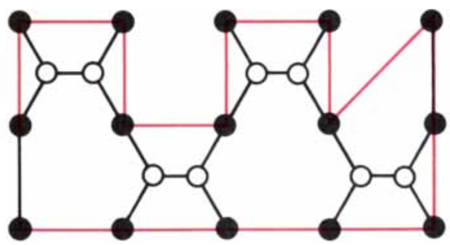


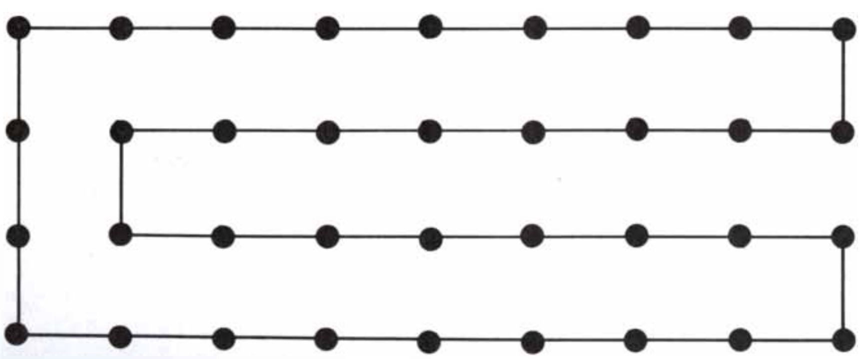


Steiner problems for the vertexes of a square and a regular pentagon, solved by soap film

Two stable soap-film trees for four points on a rectangle. Only the left tree is minimal



Path of a traveling salesman (color) *and a Steiner tree* (black) *on a rectangular lattice*



Another traveling salesman's path. Is there a Steiner tree shorter than 32.095... units?

points to be spanned in a given network. Drill holes in the sheets, insert the rods and immerse the entire assembly in a soap solution of the kind used for making bubbles. When the assembly is lifted out of the solution, a soap film forms surfaces that span the rods. Because such surfaces shrink to minimal area, the pattern formed by the film when it is viewed from above is a Steiner tree.

Such a device can find the minimal Steiner tree for the corners of a square [*see top illustration at left*]. The tree can take either of two forms, one of them a 90-degree rotation of the other. By blowing on the film you can make it jump from one pattern to its rotated form. Similarly, the device can model the minimal Steiner tree for the five points at the corners of a regular pentagon. For the six corners of a regular hexagon (and all higher regular polygons) extra Steiner points are of no help. The minimal spanning network is simply the perimeter of the polygon with one edge removed.

Even in these simple cases, however, one must be wary of the soap-film computer. For example, if the four points in the given network mark the corners of a rectangle a trifle wider than it is high, the film can stabilize in one of two patterns [*see second illustration at left*]. Both are Steiner trees, but only the one at the left is minimal. As the rectangle widens, the vertical line *AB* in the nonminimal pattern on the right becomes shorter. The line shrinks to a point when the vertical side of the rectangle is 1 and the base is $\sqrt{3}$, and for all wider rectangles only the minimal Steiner tree is stable. The tree at the right is said to be locally minimal. In other words, if you think of the lines as being elastic bands anchored at their ends to the four corner pegs, any slight shifting of the extra points will increase the length of the tree.

Given the simplicity of Kruskal's greedy algorithm for the construction of minimal spanning trees, one might suppose there would be correspondingly simple algorithms for finding minimal Steiner trees. Such, alas, is not the case. The task is part of a special class of "hard" problems known in computer science as NP-complete. When the number of points in a network is small, there are known algorithms for finding Steiner trees in a reasonably short time. As the number of points grows, however, the computing time needed increases at a rapidly accelerating pace. Even for a relatively small number of points it can be thousands or even millions of years. Most mathematicians believe no efficient al-

gorithm exists for constructing minimal Steiner trees that connect arbitrary points in the plane.

Imagine, however, that the points are arranged in a regular lattice of unit squares, like the points at the corners of the cells of a checkerboard. Is there a "good" algorithm for finding a minimal Steiner tree spanning the points of such regular patterns?

The question occurred to me several years ago when I thought of the following problem. What is the length of the minimal Steiner tree that joins the 81 points at the corners of a standard checkerboard? Henry Ernest Dudeney, England's greatest puzzle maker, and his American counterpart Sam Loyd were both fond of puzzles based on checkerboard patterns. I checked all their books carefully, but they had not considered the problem. Indeed, I could find no evidence it had ever been posed before, let alone solved.

When I tried to solve the problem, I was surprised by its complexity. Although I could not prove it, it seemed obvious that the minimal Steiner tree would be constructed by joining many replicas of the regular four-point tree. The four-point tree has no name; let us call it *X* because in working on Steiner-tree problems for rectangular lattices, an *X* is easier to draw than the full tree. The difficulty in solving such problems is that it is hard to know where to place the *X*'s. It is easy to place them so as to make a Steiner tree, but it is not so easy to make the tree minimal.

I finally convinced myself that the checkerboard puzzle has a unique answer, although I could not prove it [*see bottom illustration on these two pages*]. I call it the conjectured solution for the order-9 array, where the order is the number of points on the side of the square. Because the length of the line segments that make up each *X* is $1 + \sqrt{3}$, it is easy to determine the total length of the tree: $26\sqrt{3} + 28$, or about 73.033. Although it seemed I had found a new puzzle, I suspected that in the growing mathematical literature on Steiner trees there must surely be a paper describing a simple algorithm for finding minimal Steiner trees on rectangular lattices. I was encouraged by knowing that many problems involving paths through points in the plane, which are hard when the points are arbitrary, become trivial when the points form regular lattices.

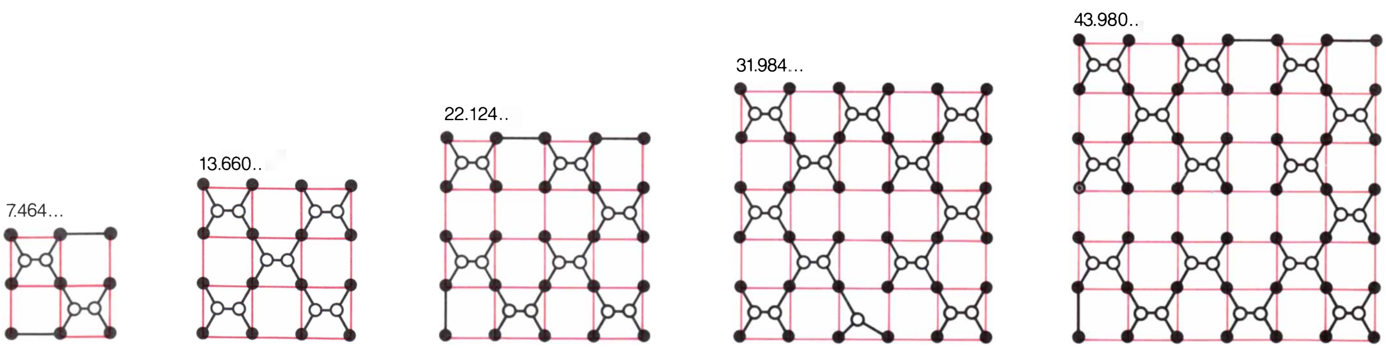
The traveling salesman problem is a notorious example. What is the shortest path allowing a salesman to visit each of n towns once and only once and return to the starting town? When the points are arbitrary, the task is NP-complete, and no efficient algorithm for solving it is known. But when the points are placed at the corners of squares and packed into a rectangular lattice, the problem is absurdly easy. If a rectangular array of m-by-n points includes an even number of points, the minimal path has length $m \times n$. If the array includes an odd number of points, the path has length $m \times n + \sqrt{2} - 1$ [*see third and bottom illustrations on page 20*]. I fully expected that the task of spanning points in such arrays by minimal Steiner trees would be equally trivial. I could not have been more wrong.

My first step was to send the checkerboard problem to my friend Ronald L. Graham, a distinguished mathematician at Bell Laboratories. I also asked him to direct me to a paper that might answer such questions. To my amazement, it turned out that the only relevant paper was one Graham himself had coauthored in 1978 with Fan R. K. Chung, also of Bell Laboratories. Titled "Steiner Trees for Ladders," it showed how to construct minimal Steiner trees for 2-by-n rectangular arrays of points, as well as for other kinds of 2-by-n "ladders." Aside from these special cases, nothing seemed to be known about how to find minimal Steiner trees for rectangular arrays when the number of points on each side is greater than 2.

The more Graham and Chung considered the matter, the more it intrigued them. On and off for better than a year they have been seeking an algorithm for the general case, but without success. Chung has recently been lecturing on the topic, and she and Graham plan eventually to write a paper on their progress.

Their best results are shown along with my checkerboard solution at the bottom of these two pages. Some of the trees have more than one minimal solution. Incredibly, only the pattern for the order-2 square lattice has been proved to be minimal. (There is a proof in Problem 73 of the book *100 Problems in Elementary Mathematics,* by Hugo Steinhaus.) Even the seemingly trivial order-3 pattern has eluded proof, although it would yield to brute-force methods carried out by computer. Graham and Chung firmly believe all their trees are minimal, but in the absence of proofs there may still be room for improvements.

It would be interesting to know whether soap film will solve the square lattices of order 3 and order 4. If it does, how far up the scale will soap film continue to find minimal trees?

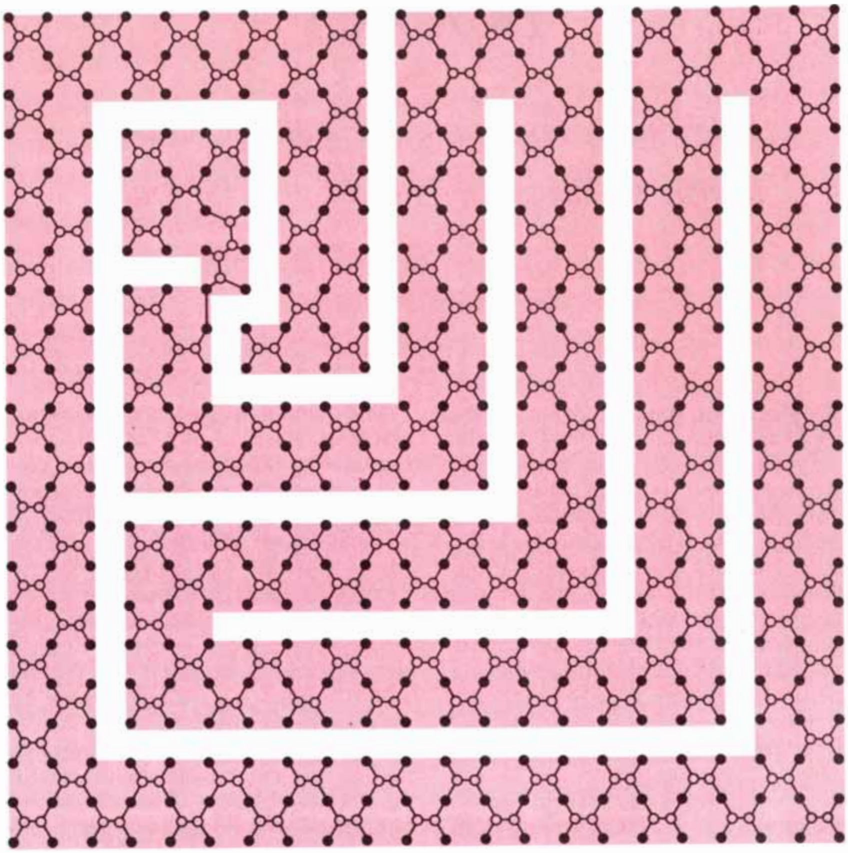


A forest of conjectured minimal Steiner trees and their lengths for low-order square lattices.

What happens when Plexiglas sheets, joined by 81 rods in the checkerboard pattern, are dipped into the soap solution and then lifted out? Will the film generate Steiner trees spanning all 81 rods? If it does, what is the probability the tree will be minimal? Perhaps some venturesome readers will carry out these experiments.

The order-6 square lattice is the smallest one from which an unexpected solution springs. When I worked on this forest of trees (sets of disconnected trees are known as forests to graph theorists), my order-6 pattern had length $11\sqrt{3} + 13$, or about 32.053. I almost fell out of my chair when I saw the shorter tree found by Graham and Chung. The little three-point tree in their pattern has length $(1 + \sqrt{3})/\sqrt{2}$, and so the total length of their network is $[(1 + \sqrt{3})/\sqrt{2}] + [11 \times (1 + \sqrt{3})]$, or about 31.984. It beautifully illustrates the kind of surprises—the "hue peculiar" of Cowper's epigraph—that lie in wait for anyone who tries to climb the ladder of square arrays in search of minimal solutions.

If you look closely, you will note that only squares of orders that are powers of 2 (2, 4, 8 and so on) have trees made entirely of X's. Graham and Chung have proved an even more general result: a rectangular array can be spanned by a Steiner tree made up entirely of X's if and only if the array is a square and the order of the square is a power of 2. Their clever proof, based on mathematical induction, is still unpublished. The unique spanning pattern generalizes in an obvious way to all squares whose order is a higher power of 2.

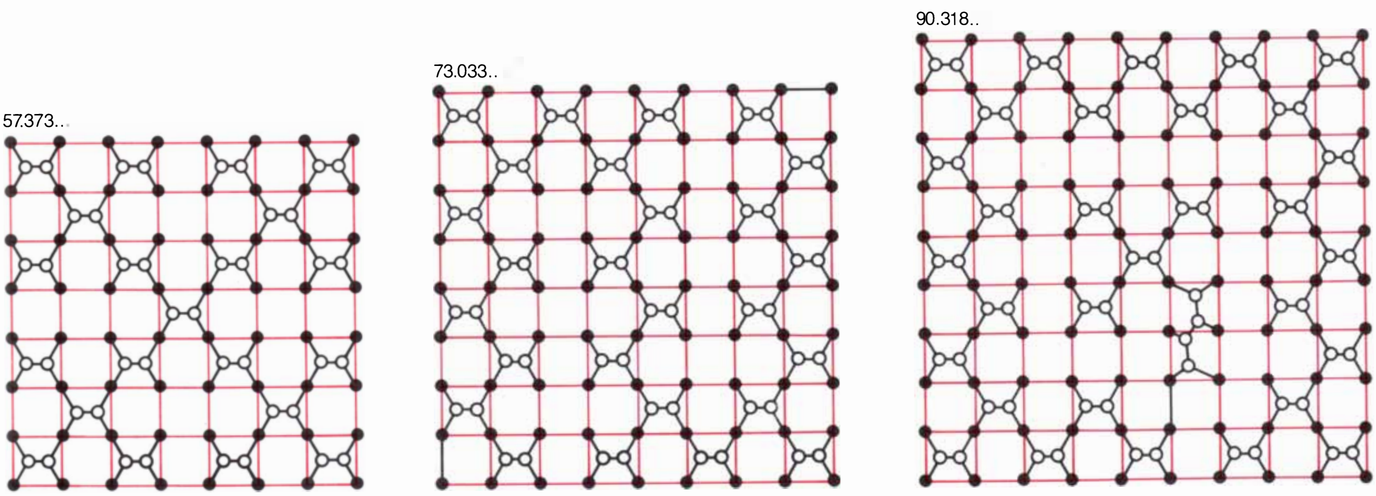


A Steiner tree of length 440.021... for the order-22 square lattice. Is it minimal?

Space does not allow me to provide examples of the best-known patterns for nonsquare rectangular arrays, for which Graham and Chung have many curious results and conjectures. I close by giving the best Steiner tree they have found for the order-22 square [*see illustration above*]. It includes a pattern bounded by six points on two squares, which does not match the familiar X. The six-point pattern also appears in the Steiner tree of order 10, and its length is

$$\sqrt{11 + 6\sqrt{3}},$$

or about 4.625. The total length of the tree is approximately 440.021. If any reader can shorten that length, please let me know.



The author's solution to the checkerboard problem is shown on the order-9 square lattice